Rank Constrained Homotopies

Kaushika De Silva

Department of Mathematics, Purdue University

42nd Canadian Operator Algebras Symposium
June 23rd, 2014

Notation and Statement of the Main Theorem

Notation

- X: Compact Hausdorff space of finite covering dimension.
- $(M_n)_+$: non-negative definite $n \times n$ matrices over $\mathbb{C}$
- $S(n, k, l) = \{b \in (M_n)_+ | \, l \leq \text{rank}(b) \leq k\}$
- $F(X, S(n, k, l)) = \{f : X \to S(n, k, l) : f \text{ is continuous}\}$

Theorem

For any $n, k, l \in \mathbb{N}$, if $\lfloor \frac{\dim X}{2} \rfloor \leq k - l$, then $F(X, S(n, k, l))$ is path connected. In particular, $\forall n, k, l \in \mathbb{N}$, $\pi_r(S(n, k, l)) = 0$, whenever $r \leq 2(k - l) + 1$.

Notation and Statement of the Main Theorem

Notation

- X: Compact Hausdorff space of finite covering dimension.
- $(M_n)_+$: non-negative definite $n \times n$ matrices over $\mathbb{C}$
- $S(n, k, l) = \{b \in (M_n)_+ | l \leq \text{rank}(b) \leq k\}$
- $F(X, S(n, k, l)) = \{f : X \to S(n, k, l) : f \text{ is continuous}\}$

Theorem

For any $n, k, l \in \mathbb{N}$, if $\lfloor \frac{\dim X}{2} \rfloor \leq k - l$, then $F(X, S(n, k, l))$ is path connected. In particular, $\forall n, k, l \in \mathbb{N}$, $\pi_r(S(n, k, l)) = 0$, whenever $r \leq 2(k - l) + 1$.

Notation and Statement of the Main Theorem

Notation

- X: Compact Hausdorff space of finite covering dimension.
- $(M_n)_+$: non-negative definite $n \times n$ matrices over $\mathbb{C}$
- $S(n, k, l) = \{b \in (M_n)_+ | \, l \leq \text{rank}(b) \leq k\}$
- $F(X, S(n, k, l)) = \{f \colon X \to S(n, k, l) \colon f \text{ is continuous}\}$

Theorem

For any $n, k, l \in \mathbb{N}$, if $\lfloor \frac{\dim X}{2} \rfloor \leq k - l$, then $F(X, S(n, k, l))$ is path connected. In particular, $\forall n, k, l \in \mathbb{N}$, $\pi_r(S(n, k, l)) = 0$, whenever $r \leq 2(k - l) + 1$.

Notation and Statement of the Main Theorem

Notation

- X: Compact Hausdorff space of finite covering dimension.
- $(M_n)_+$: non-negative definite $n \times n$ matrices over $\mathbb{C}$
- $S(n, k, l) = \{b \in (M_n)_+ | \, l \leq \text{rank}(b) \leq k\}$
- $F(X, S(n, k, l)) = \{f : X \to S(n, k, l) : f \text{ is continuous}\}$

Theorem

For any $n, k, l \in \mathbb{N}$, if $\lfloor \frac{\dim X}{2} \rfloor \leq k - l$, then $F(X, S(n, k, l))$ is path connected. In particular, $\forall n, k, l \in \mathbb{N}$, $\pi_r(S(n, k, l)) = 0$, whenever $r \leq 2(k - l) + 1$.

Notation and Statement of the Main Theorem

Notation

- X: Compact Hausdorff space of finite covering dimension.
- $(M_n)_+$: non-negative definite $n \times n$ matrices over $\mathbb{C}$
- $S(n, k, l) = \{b \in (M_n)_+ | \, l \leq \text{rank}(b) \leq k\}$
- $F(X, S(n, k, l)) = \{f \colon X \to S(n, k, l) \colon f \text{ is continuous}\}$

Theorem

For any $n, k, l \in \mathbb{N}$, if $\lfloor \frac{\dim X}{2} \rfloor \leq k - l$, then $F(X, S(n, k, l))$ is path connected. In particular, $\forall n, k, l \in \mathbb{N}$, $\pi_r(S(n, k, l)) = 0$, whenever $r \leq 2(k - l) + 1$.

Motivation

- Homotopy properties of the space $S(n, k, l)$ has applications in C^*-algebra theory.
- Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, where $T(A)$ is the tracial state space of A.
- Any $a \in A_+$, induce a lower semi-continuous affine function ι_a on $T(A)$, given by,

$$\iota_a(\tau) = \lim_n(\tau(a^{1/n}))$$

- If K denote the compacts on a separable Hilbert space, ι_a extends to $(A \otimes K)_+$ in a natural way.

Motivation

- Homotopy properties of the space $S(n,k,l)$ has applications in C^*-algebra theory.
- Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, where $T(A)$ is the tracial state space of A.
- Any $a \in A_+$, induce a lower semi-continuous affine function ι_a on $T(A)$, given by,

$$\iota_a(\tau) = \lim_n (\tau(a^{1/n}))$$

- If K denote the compacts on a separable Hilbert space, ι_a extends to $(A \otimes K)_+$ in a natural way.

Motivation

- Homotopy properties of the space $S(n,k,l)$ has applications in C^*-algebra theory.
- Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, where $T(A)$ is the tracial state space of A.
- Any $a \in A_+$, induce a lower semi-continuous affine function ι_a on $T(A)$, given by,

$$\iota_a(\tau) = \lim_n (\tau(a^{1/n}))$$

- If K denote the compacts on a separable Hilbert space, ι_a extends to $(A \otimes K)_+$ in a natural way.

Motivation

- Homotopy properties of the space $S(n,k,l)$ has applications in C^*-algebra theory.
- Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, where $T(A)$ is the tracial state space of A.
- Any $a \in A_+$, induce a lower semi-continuous affine function ι_a on $T(A)$, given by,

$$\iota_a(\tau) = \lim_n (\tau(a^{1/n}))$$

- If K denote the compacts on a separable Hilbert space, ι_a extends to $(A \otimes K)_+$ in a natural way.

Motivation

- Question ?
  For a unital, simple C^*-algebra A, is it possible to approximate strictly positive continuous affine functions ($Aff(T(A))_{++}$) by functions of the form ι_a, $a \in (A \otimes K)_+$?
- (Andrew Toms, 2009), For (unital, simple) *ASH* algebras with slow dimension growth, the question has a positive answer.
- Following is a key proposition in the proof of the above.

Lemma (Toms)

For any $n, k, l \in \mathbb{N}$, if $4\dim X \leq k - l$, then $F(X, S(n, k, l))$ is path connected.

Motivation

- Question ?
  For a unital, simple C^*-algebra A, is it possible to approximate strictly positive continuous affine functions ($Aff(T(A))_{++}$) by functions of the form ι_a, $a \in (A \otimes K)_+$?
- (Andrew Toms, 2009), For (unital, simple) *ASH* algebras with slow dimension growth, the question has a positive answer.
- Following is a key proposition in the proof of the above.

Lemma (Toms)

For any $n, k, l \in \mathbb{N}$, if $4 \dim X \leq k - l$, then $F(X, S(n, k, l))$ is path connected.

Motivation

- Question ?
  For a unital, simple C^*-algebra A, is it possible to approximate strictly positive continuous affine functions $(Aff(T(A))_{++})$ by functions of the form ι_a, $a \in (A \otimes K)_+$?
- (Andrew Toms, 2009), For (unital, simple) *ASH* algebras with slow dimension growth, the question has a positive answer.
- Following is a key proposition in the proof of the above.

Lemma (Toms)

For any $n, k, l \in \mathbb{N}$, if $4 \dim X \leq k - l$, then $F(X, S(n, k, l))$ is path connected.

Motivation

- Question ?
  For a unital, simple C^*-algebra A, is it possible to approximate strictly positive continuous affine functions ($Aff(T(A))_{++}$) by functions of the form ι_a, $a \in (A \otimes K)_+$?
- (Andrew Toms, 2009), For (unital, simple) *ASH* algebras with slow dimension growth, the question has a positive answer.
- Following is a key proposition in the proof of the above.

Lemma (Toms)

For any $n, k, l \in \mathbb{N}$, if $4 \dim X \leq k - l$, then $F(X, S(n,k,l))$ is path connected.

Recap from Vector bundle theory

Recall...

- A (complex) vector bundle over X is a triple (E, p, X), where E is a topological space, $p\colon E \to X$ is a continuous map with each fiber $p^{-1}(x) = E_x$ admitting a $\mathbb{C}$-vector space structure.
- Let $\theta^k(X) = (X \times \mathbb{C}^k, \pi, X)$, where π is the projection onto X. θ^k is called the k-dimensional product bundle.
- $\alpha = (E, p, X)$ is called trivial if $\alpha \cong \theta^k(X)$ for some k.

Definition

$\alpha = (E, p, X)$ is locally trivial if X has an open covering $\{U_\lambda\}$ such that $\alpha|_{U_\lambda} \cong \theta^k(U_\lambda)$ for each λ.

Recap from Vector bundle theory

Recall...

- A (complex) vector bundle over X is a triple (E, p, X), where E is a topological space, $p\colon E \to X$ is a continuous map with each fiber $p^{-1}(x) = E_x$ admitting a $\mathbb{C}$-vector space structure.
- Let $\theta^k(X) = (X \times \mathbb{C}^k, \pi, X)$, where π is the projection onto X. θ^k is called the k-dimensional product bundle.
- $\alpha = (E, p, X)$ is called trivial if $\alpha \cong \theta^k(X)$ for some k.

Definition

$\alpha = (E, p, X)$ is locally trivial if X has an open covering $\{U_\lambda\}$ such that $\alpha|_{U_\lambda} \cong \theta^k(U_\lambda)$ for each λ.

Recap from Vector bundle theory

Recall...

- A (complex) vector bundle over X is a triple (E, p, X), where E is a topological space, $p\colon E \to X$ is a continuous map with each fiber $p^{-1}(x) = E_x$ admitting a $\mathbb{C}$-vector space structure.
- Let $\theta^k(X) = (X \times \mathbb{C}^k, \pi, X)$, where π is the projection onto X. θ^k is called the k-dimensional product bundle.
- $\alpha = (E, p, X)$ is called trivial if $\alpha \cong \theta^k(X)$ for some k.

Definition

$\alpha = (E, p, X)$ is locally trivial if X has an open covering $\{U_\lambda\}$ such that $\alpha|_{U_\lambda} \cong \theta^k(U_\lambda)$ for each λ.

Recap from Vector bundle theory

Recall...

- A (complex) vector bundle over X is a triple (E, p, X), where E is a topological space, $p\colon E \to X$ is a continuous map with each fiber $p^{-1}(x) = E_x$ admitting a $\mathbb{C}$-vector space structure.
- Let $\theta^k(X) = (X \times \mathbb{C}^k, \pi, X)$, where π is the projection onto X. θ^k is called the k-dimensional product bundle.
- $\alpha = (E, p, X)$ is called trivial if $\alpha \cong \theta^k(X)$ for some k.

Definition

$\alpha = (E, p, X)$ is locally trivial if X has an open covering $\{U_\lambda\}$ such that $\alpha|_{U_\lambda} \cong \theta^k(U_\lambda)$ for each λ.

Stability properties of locally trivial bundles

- Let $\mathbf{Bun}_k(X)$ denote the category of all locally trivial (Complex) vector bundles over X, of dimension k.

Theorem

Let X be a (para) compact, Hausdorff and finite dimensional topological space.

- *If $\alpha \in \mathbf{Bun}_k(X)$, then there is a trivial vector bundle δ over X with $\dim \delta \geq k - \lfloor \frac{\dim X}{2} \rfloor$, such that δ is a direct summand of α. i.e. $\alpha = \delta \oplus \eta$, for some bundle η*
- *Let $\alpha, \beta \in \mathbf{Bun}_k(X)$. If $k \geq \frac{\dim X}{2}$ and $\alpha \oplus \gamma \cong \beta \oplus \gamma$ for some bundle γ over X, then $\alpha \cong \beta$.*

Stability properties of locally trivial bundles

- Let $\mathbf{Bun}_k(X)$ denote the category of all locally trivial (Complex) vector bundles over X, of dimension k.

Theorem

Let X be a (para) compact, Hausdorff and finite dimensional topological space.

- *If $\alpha \in \mathbf{Bun}_k(X)$, then there is a trivial vector bundle δ over X with $\dim \delta \geq k - \lfloor \frac{\dim X}{2} \rfloor$, such that δ is a direct summand of α. i.e. $\alpha = \delta \oplus \eta$, for some bundle η*
- *Let $\alpha, \beta \in \mathbf{Bun}_k(X)$. If $k \geq \frac{\dim X}{2}$ and $\alpha \oplus \gamma \cong \beta \oplus \gamma$ for some bundle γ over X, then $\alpha \cong \beta$.*

Stability properties of locally trivial bundles

- Let $\mathbf{Bun}_k(X)$ denote the category of all locally trivial (Complex) vector bundles over X, of dimension k.

Theorem

Let X be a (para) compact, Hausdorff and finite dimensional topological space.

- *If $\alpha \in \mathbf{Bun}_k(X)$, then there is a trivial vector bundle δ over X with $\dim \delta \geq k - \lfloor \frac{\dim X}{2} \rfloor$, such that δ is a direct summand of α. i.e. $\alpha = \delta \oplus \eta$, for some bundle η*
- *Let $\alpha, \beta \in \mathbf{Bun}_k(X)$. If $k \geq \frac{\dim X}{2}$ and $\alpha \oplus \gamma \cong \beta \oplus \gamma$ for some bundle γ over X, then $\alpha \cong \beta$.*

Stability properties of locally trivial bundles

- Let $\mathbf{Bun}_k(X)$ denote the category of all locally trivial (Complex) vector bundles over X, of dimension k.

Theorem

Let X be a (para) compact, Hausdorff and finite dimensional topological space.

- *If $\alpha \in \mathbf{Bun}_k(X)$, then there is a trivial vector bundle δ over X with $\dim \delta \geq k - \lfloor \frac{\dim X}{2} \rfloor$, such that δ is a direct summand of α. i.e. $\alpha = \delta \oplus \eta$, for some bundle η*
- *Let $\alpha, \beta \in \mathbf{Bun}_k(X)$. If $k \geq \frac{\dim X}{2}$ and $\alpha \oplus \gamma \cong \beta \oplus \gamma$ for some bundle γ over X, then $\alpha \cong \beta$.*

Projections and Vector Bundles

- Given a projection $p \in M_n(C(X))$, there is an associated vector bundle $\epsilon(p) = (E_p, \pi, X)$, with

$$E(p) = \{(x, v) \colon x \in X, v \in p(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n.$$

- Moreover, every locally trivial vector bundle over X can be realized in the above form [R. G. Swan, 1961].
- That is, fixed $n \geq k + \frac{dim X}{2}$, for each $\alpha \in \mathbf{Bun}_k(X)$ there is a projection $p_\alpha \in M_n(C(X))$ such that $\epsilon(p_\alpha) \cong \alpha$.
- Moreover, $\alpha \cong \beta$ iff $p_\alpha \sim_{M.V} p_\beta$.

Projections and Vector Bundles

- Given a projection $p \in M_n(C(X))$, there is an associated vector bundle $\epsilon(p) = (E_p, \pi, X)$, with

$$E(p) = \{(x, v) \colon x \in X, v \in p(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n.$$

- Moreover, every locally trivial vector bundle over X can be realized in the above form [R. G. Swan, 1961].
- That is, fixed $n \geq k + \frac{dim X}{2}$, for each $\alpha \in \mathbf{Bun}_k(X)$ there is a projection $p_\alpha \in M_n(C(X))$ such that $\epsilon(p_\alpha) \cong \alpha$.
- Moreover, $\alpha \cong \beta$ iff $p_\alpha \sim_{M.V} p_\beta$.

Projections and Vector Bundles

- Given a projection $p \in M_n(C(X))$, there is an associated vector bundle $\epsilon(p) = (E_p, \pi, X)$, with

$$E(p) = \{(x, v) \colon x \in X, v \in p(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n.$$

- Moreover, every locally trivial vector bundle over X can be realized in the above form [R. G. Swan, 1961].
- That is, fixed $n \geq k + \frac{dim X}{2}$, for each $\alpha \in \mathbf{Bun}_k(X)$ there is a projection $p_\alpha \in M_n(C(X))$ such that $\epsilon(p_\alpha) \cong \alpha$.
- Moreover, $\alpha \cong \beta$ iff $p_\alpha \sim_{M.V} p_\beta$.

Projections and Vector Bundles

- Given a projection $p \in M_n(C(X))$, there is an associated vector bundle $\epsilon(p) = (E_p, \pi, X)$, with

$$E(p) = \{(x, v) \colon x \in X, v \in p(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n.$$

- Moreover, every locally trivial vector bundle over X can be realized in the above form [R. G. Swan, 1961].
- That is, fixed $n \geq k + \frac{dim X}{2}$, for each $\alpha \in \mathbf{Bun}_k(X)$ there is a projection $p_\alpha \in M_n(C(X))$ such that $\epsilon(p_\alpha) \cong \alpha$.
- Moreover, $\alpha \cong \beta$ iff $p_\alpha \sim_{M.V} p_\beta$.

Outline of the proof

- Given $a, b \in F(X, S(n, k, l))$ with $k - l \geq \lfloor \frac{dim\, X}{2} \rfloor$, need to show that $\exists\, h\colon [0, 1] \to F(X, (n, k, l))$ with $h(0) = a$, $h(1) = b$.

Lemma (A)

If $k - l \geq \lfloor \frac{dim\, X}{2} \rfloor$, then $\forall a \in F(X, S(n, k, l))$, $\exists\, p \in M_n(C(X))$ a trivial projection of rank l such that

$$dim[(p(x) + a(x))(\mathbb{C}^n)] \leq k, \forall x \in X$$

Lemma (B)

Suppose $n \geq l + \frac{dim\, X}{2}$ and let $p, q \in M_n(C(X))$ be trivial projections of rank l. Then $\exists\, h\colon [0, 1] \to Proj(M_n(C(X)))$ with $h(0) = p$ and $h(1) = q$.

Outline of the proof

- Given $a, b \in F(X, S(n, k, l))$ with $k - l \geq \lfloor \frac{dim X}{2} \rfloor$, need to show that $\exists$ $h\colon [0, 1] \to F(X, (n, k, l))$ with $h(0) = a$, $h(1) = b$.

Lemma (A)

If $k - l \geq \lfloor \frac{dim X}{2} \rfloor$, then $\forall a \in F(X, S(n, k, l))$, $\exists$ $p \in M_n(C(X))$ a trivial projection of rank l such that

$$dim[(p(x) + a(x))(\mathbb{C}^n)] \leq k, \forall x \in X$$

Lemma (B)

Suppose $n \geq l + \frac{dim X}{2}$ and let $p, q \in M_n(C(X))$ be trivial projections of rank l. Then $\exists$ $h\colon [0, 1] \to Proj(M_n(C(X)))$ with $h(0) = p$ and $h(1) = q$.

Outline of the proof

- Given $a, b \in F(X, S(n, k, l))$ with $k - l \geq \lfloor \frac{dim X}{2} \rfloor$, need to show that $\exists\ h\colon [0, 1] \to F(X, (n, k, l))$ with $h(0) = a$, $h(1) = b$.

Lemma (A)

If $k - l \geq \lfloor \frac{dim X}{2} \rfloor$, then $\forall a \in F(X, S(n, k, l))$, $\exists\ p \in M_n(C(X))$ a trivial projection of rank l such that

$$dim[(p(x) + a(x))(\mathbb{C}^n)] \leq k, \forall x \in X$$

Lemma (B)

Suppose $n \geq l + \frac{dim X}{2}$ and let $p, q \in M_n(C(X))$ be trivial projections of rank l. Then $\exists\ h\colon [0, 1] \to Proj(M_n(C(X)))$ with $h(0) = p$ and $h(1) = q$.

Idea of the proof of Lemma A

- Given $a \in F(X, S(n, k, l))$ there associates a vector bundle $\epsilon(a) = (E(a), p, X)$. Here,

$$E(a) = \{(x, v) \colon x \in X, v \in a(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n$$

  and p is the restriction of canonical projection of $X \times \mathbb{C}^n$ onto X, to $E(a)$. The bundle $\epsilon(a)$ is not necessarily locally trivial.
- However, we can partition X so that the restriction $\epsilon(a)$ to each set in the partition is a locally trivial bundle.
- Then, to get the trivial projection given in the conclusion of Lemma A, we apply stability properties of locally trivial bundles discussed before, to restricted bundles.
- To define the trivial projection globally we will use some extension results due to Chris Phillips and Andrew Toms.

Idea of the proof of Lemma A

- Given $a \in F(X, S(n,k,l))$ there associates a vector bundle $\epsilon(a) = (E(a), p, X)$. Here,

$$E(a) = \{(x, v) \colon x \in X, v \in a(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n$$

  and p is the restriction of canonical projection of $X \times \mathbb{C}^n$ onto X, to $E(a)$. The bundle $\epsilon(a)$ is not necessarily locally trivial.
- However, we can partition X so that the restriction $\epsilon(a)$ to each set in the partition is a locally trivial bundle.
- Then, to get the trivial projection given in the conclusion of Lemma A, we apply stability properties of locally trivial bundles discussed before, to restricted bundles.
- To define the trivial projection globally we will use some extension results due to Chris Phillips and Andrew Toms.

Idea of the proof of Lemma A

- Given $a \in F(X, S(n, k, l))$ there associates a vector bundle $\epsilon(a) = (E(a), p, X)$. Here,

$$E(a) = \{(x, v)\colon x \in X, v \in a(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n$$

  and p is the restriction of canonical projection of $X \times \mathbb{C}^n$ onto X, to $E(a)$. The bundle $\epsilon(a)$ is not necessarily locally trivial.
- However, we can partition X so that the restriction $\epsilon(a)$ to each set in the partition is a locally trivial bundle.
- Then, to get the trivial projection given in the conclusion of Lemma A, we apply stability properties of locally trivial bundles discussed before, to restricted bundles.
- To define the trivial projection globally we will use some extension results due to Chris Phillips and Andrew Toms.

Idea of the proof of Lemma A

- Given $a \in F(X, S(n,k,l))$ there associates a vector bundle $\epsilon(a) = (E(a), p, X)$. Here,
$$E(a) = \{(x, v) \colon x \in X, v \in a(x)(\mathbb{C}^n)\} \subset X \times \mathbb{C}^n$$
and p is the restriction of canonical projection of $X \times \mathbb{C}^n$ onto X, to $E(a)$. The bundle $\epsilon(a)$ is not necessarily locally trivial.
- However, we can partition X so that the restriction $\epsilon(a)$ to each set in the partition is a locally trivial bundle.
- Then, to get the trivial projection given in the conclusion of Lemma A, we apply stability properties of locally trivial bundles discussed before, to restricted bundles.
- To define the trivial projection globally we will use some extension results due to Chris Phillips and Andrew Toms.

Idea of the proof of Lemma A

- Let $a \in F(X, S(n, k, l))$ and suppose the rank values of a are $n_1 < n_2 < .... < n_L$. For simplicity let $\epsilon = \epsilon(a)$.
- For $1 \leq i \leq L$, set $E_i = \{x \in X : \text{rank } a(x) = n_i\}$. Then $\epsilon|_{E_i}$ is locally trivial. The support projection of $a|_{E_i}$ is continuous and $\epsilon|_{E_i}$ is the bundle corresponding to this projection.

Definition (Toms)

A positive element $a \in M_n(C(X))_+$ is *well-supported*, if $\forall 1 \leq i \leq L, \exists p_i : \overline{E}_i \to Proj(M_n)$ such that for each i,

$$p_i(x) = \lim_n (a(x))^{1/n}, \forall x \in E_i \text{ and}$$

for each pair (i, j) with $j \geq i$,

$$p_i(x) \leq p_j(x), \forall x \in \overline{E}_i \cap \overline{E}_j$$

Idea of the proof of Lemma A

- Let $a \in F(X, S(n,k,l))$ and suppose the rank values of a are $n_1 < n_2 < .... < n_L$. For simplicity let $\epsilon = \epsilon(a)$.
- For $1 \leq i \leq L$, set $E_i = \{x \in X : \text{rank } a(x) = n_i\}$. Then $\epsilon|_{E_i}$ is locally trivial. The support projection of $a|_{E_i}$ is continuous and $\epsilon|_{E_i}$ is the bundle corresponding to this projection.

Definition (Toms)

A positive element $a \in M_n(C(X))_+$ is *well-supported*, if $\forall 1 \leq i \leq L, \exists\, p_i : \overline{E}_i \to Proj(M_n)$ such that for each i,

$$p_i(x) = \lim_n (a(x))^{1/n}, \forall x \in E_i \text{ and}$$

for each pair (i, l) with $j \geq i$,

$$p_i(x) \leq p_j(x), \forall x \in \overline{E}_i \cap \overline{E}_j$$

Idea of the proof of Lemma A

- Let $a \in F(X, S(n,k,l))$ and suppose the rank values of a are $n_1 < n_2 < .... < n_L$. For simplicity let $\epsilon = \epsilon(a)$.
- For $1 \leq i \leq L$, set $E_i = \{x \in X \colon \text{rank } a(x) = n_i\}$. Then $\epsilon|_{E_i}$ is locally trivial. The support projection of $a|_{E_i}$ is continuous and $\epsilon|_{E_i}$ is the bundle corresponding to this projection.

Definition (Toms)

A positive element $a \in M_n(C(X))_+$ is *well-supported*, if $\forall\, 1 \leq i \leq L$, $\exists\, p_i \colon \overline{E}_i \to Proj(M_n)$ such that for each i,

$$p_i(x) = \lim_n (a(x))^{1/n}, \forall\, x \in E_i \text{ and}$$

for each pair (i, l) with $j \geq i$,

$$p_i(x) \leq p_j(x), \forall\, x \in \overline{E}_i \cap \overline{E}_j$$

Idea of the proof of Lemma A

Theorem (Toms)

Let $a \in M_n(C(X))_+$ then there exists a well-supported $b \in M_n(C(X))_+$ such that the set of rank values of b is same as the set of rank values of a and $b \leq a$.

Corollary

Given $a \in F(X, S(n, k, l))$, there is $b \in F(X, S(n, k, l))$ with b homotopic to a.

- Corollary reduces the proof of Lemma A to the case of $a \in F(X, S(n, k, l))$ being well- supported.

Idea of the proof of Lemma A

Theorem (Toms)

Let $a \in M_n(C(X))_+$ then there exists a well-supported $b \in M_n(C(X))_+$ such that the set of rank values of b is same as the set of rank values of a and $b \leq a$.

Corollary

Given $a \in F(X, S(n, k, l))$, there is $b \in F(X, S(n, k, l))$ with b homotopic to a.

- Corollary reduces the proof of Lemma A to the case of $a \in F(X, S(n, k, l))$ being well- supported.

Idea of the proof of Lemma A

Theorem (Toms)

Let $a \in M_n(C(X))_+$ then there exists a well-supported $b \in M_n(C(X))_+$ such that the set of rank values of b is same as the set of rank values of a and $b \leq a$.

Corollary

Given $a \in F(X, S(n, k, l))$, there is $b \in F(X, S(n, k, l))$ with b homotopic to a.

- Corollary reduces the proof of Lemma A to the case of $a \in F(X, S(n, k, l))$ being well- supported.

Proof of Lemma A

- Let $a \in F(X, S(n, k, l))$ be well supported and choose $E_1, E_2, ...E_L$ and $p_1, p_2, ...p_L$ be as given by the definition of well-supportedness. Write $F_i = \overline{E}_i$.
- Choose a trivial projection $q_1 \in M_n(C(F_1))$, with
$$\text{rank } q_1 = n_1 - \left\lfloor \frac{d}{2} \right\rfloor \text{ and } q_1 \leq p_1.$$
- By Corollaries 1 and 2, extend q to a trivial projection $q_1 \in M_n(C(X))$ such that $\forall 1 \leq i \leq L$,
$$q_1(x) \leq p_i(x), \forall x \in F_i.$$
- Since q_1 is trivial and $n - \text{rank } q_1 \geq k - l \geq \lfloor \frac{d}{2} \rfloor$, $q_1{}^{\perp} = 1_n - q_1$ is also trivial and hence
$$q_1{}^{\perp} M_n(C(X)) q_1{}^{\perp} \cong M_{n-n_1+\lfloor \frac{d}{2} \rfloor}(C(X))$$

Proof of Lemma A

- Let $a \in F(X, S(n,k,l))$ be well supported and choose $E_1, E_2, ...E_L$ and $p_1, p_2, ...p_L$ be as given by the definition of well-supportedness. Write $F_i = \overline{E}_i$.
- Choose a trivial projection $q_1 \in M_n(C(F_1))$, with

$$\text{rank } q_1 = n_1 - \left\lfloor \frac{d}{2} \right\rfloor \text{ and } q_1 \le p_1 .$$

- By Corollaries 1 and 2, extend q to a trivial projection $q_1 \in M_n(C(X))$ such that $\forall 1 \le i \le L$,

$$q_1(x) \le p_i(x) \, , \forall x \in F_i .$$

- Since q_1 is trivial and $n - \text{rank } q_1 \ge k - l \ge \left\lfloor \frac{d}{2} \right\rfloor$, ${q_1}^{\perp} = 1_n - q_1$ is also trivial and hence

$$ {q_1}^{\perp} M_n(C(X)) {q_1}^{\perp} \cong M_{n-n_1+\left\lfloor \frac{d}{2} \right\rfloor}(C(X))$$

Proof of Lemma A

- Let $a \in F(X, S(n, k, l))$ be well supported and choose $E_1, E_2, ...E_L$ and $p_1, p_2, ...p_L$ be as given by the definition of well-supportedness. Write $F_i = \overline{E}_i$.
- Choose a trivial projection $q_1 \in M_n(C(F_1))$, with

$$\text{rank } q_1 = n_1 - \left\lfloor \frac{d}{2} \right\rfloor \text{ and } q_1 \leq p_1 .$$

- By Corollaries 1 and 2, extend q to a trivial projection $q_1 \in M_n(C(X))$ such that $\forall 1 \leq i \leq L$,

$$q_1(x) \leq p_i(x) , \forall x \in F_i .$$

- Since q_1 is trivial and $n - \text{rank } q_1 \geq k - l \geq \left\lfloor \frac{d}{2} \right\rfloor$, $q_1{}^{\perp} = 1_n - q_1$ is also trivial and hence

$$q_1{}^{\perp} M_n(C(X)) q_1{}^{\perp} \cong M_{n - n_1 + \left\lfloor \frac{d}{2} \right\rfloor}(C(X))$$

Proof of Lemma A

- Let $a \in F(X, S(n, k, l))$ be well supported and choose $E_1, E_2, ...E_L$ and $p_1, p_2, ...p_L$ be as given by the definition of well-supportedness. Write $F_i = \overline{E}_i$.
- Choose a trivial projection $q_1 \in M_n(C(F_1))$, with

$$\text{rank } q_1 = n_1 - \left\lfloor \frac{d}{2} \right\rfloor \text{ and } q_1 \leq p_1 .$$

- By Corollaries 1 and 2, extend q to a trivial projection $q_1 \in M_n(C(X))$ such that $\forall 1 \leq i \leq L$,

$$q_1(x) \leq p_i(x) , \forall x \in F_i .$$

- Since q_1 is trivial and $n - \text{rank } q_1 \geq k - l \geq \lfloor \frac{d}{2} \rfloor$, ${q_1}^{\perp} = 1_n - q_1$ is also trivial and hence

$${q_1}^{\perp} M_n(C(X)) {q_1}^{\perp} \cong M_{n-n_1+\lfloor \frac{d}{2} \rfloor}(C(X))$$

Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ and let $p, q \in M_n(C(X))$ be trivial projections of rank l. Then $\exists\ h\colon [0, 1] \to Proj(M_n(C(X)))$ with $h(0) = p$ and $h(1) = q$.

- Sketch of the proof of Lemma B:
    - Lemma holds when X is a CW-complex. (Is a consequnce of the homotopy classification theorem for bundles over CW-complexes.)
    - If X is a compact metric space, X is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha,\beta}\}$ of finite CW-complexes and we use a standard approximation argument to prove the result.
    - For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X.

Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ and let $p, q \in M_n(C(X))$ be trivial projections of rank l. Then $\exists\ h\colon [0, 1] \to Proj(M_n(C(X)))$ with $h(0) = p$ and $h(1) = q$.

- Sketch of the proof of Lemma B:
  - Lemma holds when X is a CW-complex. (Is a consequnce of the homotopy classification theorem for bundles over CW-complexes)
  - If X is a compact metric space, X is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha,\beta}\}$ of finite CW-complexes and we use a standard approximation argument to prove the result.
  - For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X.

Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ and let $p, q \in M_n(C(X))$ be trivial projections of rank l. Then $\exists\ h\colon [0, 1] \to Proj(M_n(C(X)))$ with $h(0) = p$ and $h(1) = q$.

- Sketch of the proof of Lemma B:
  - Lemma holds when X is a *CW*-complex. (Is a consequnce of the homotopy classification theorem for bundles over *CW*-complexes)
  - If X is a compact metric space, X is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha,\beta}\}$ of finite *CW*-complexes and we use a standard approximation argument to prove the result.
  - For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X.

Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ *and let* $p, q \in M_n(C(X))$ *be trivial projections of rank* l. *Then* $\exists\ h\colon [0,1] \to Proj(M_n(C(X)))$ *with* $h(0) = p$ *and* $h(1) = q$.

- Sketch of the proof of Lemma B:
  - Lemma holds when X is a CW-complex. (Is a consequnce of the homotopy classification theorem for bundles over CW-complexes)
  - If X is a compact metric space, X is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha,\beta}\}$ of finite CW-complexes and we use a standard approximation argument to prove the result.
  - For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X.

Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ *and let* $p, q \in M_n(C(X))$ *be trivial projections of rank* l*. Then* $\exists\ h\colon [0, 1] \to Proj(M_n(C(X)))$ *with* $h(0) = p$ *and* $h(1) = q$*.*

- Sketch of the proof of Lemma B:
  - Lemma holds when X is a CW-complex. (Is a consequnce of the homotopy classification theorem for bundles over CW-complexes)
  - If X is a compact metric space, X is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha,\beta}\}$ of finite CW-complexes and we use a standard approximation argument to prove the result.
  - For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X.

Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ *and let* $p, q \in M_n(C(X))$ *be trivial projections of rank* l*. Then* $\exists\ h\colon [0,1] \to Proj(M_n(C(X)))$ *with* $h(0) = p$ *and* $h(1) = q$*.*

- Sketch of the proof of Lemma B:
  - Lemma holds when X is a CW-complex. (Is a consequnce of the homotopy classification theorem for bundles over CW-complexes)
  - If X is a compact metric space, X is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha,\beta}\}$ of finite CW-complexes and we use a standard approximation argument to prove the result.
  - For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X.

Proof of Lemma B

Lemma (B)

Suppose $n \geq l + \frac{\dim X}{2}$ *and let* $p, q \in M_n(C(X))$ *be trivial projections of rank* l. *Then* $\exists\ h\colon [0,1] \to Proj(M_n(C(X)))$ *with* $h(0) = p$ *and* $h(1) = q$.

- Sketch of the proof of Lemma B:
  - Lemma holds when X is a CW-complex. (Is a consequnce of the homotopy classification theorem for bundles over CW-complexes)
  - If X is a compact metric space, X is the inverse limit of inverse system $\{X_\alpha, \psi_{\alpha,\beta}\}$ of finite CW-complexes and we use a standard approximation argument to prove the result.
  - For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X.

Continuity of path connectedness of $F(X,(S(n,k,l)))$

Theorem

Suppose for each finite simplicial complex Z with $\dim Z \leq d$, the function space $F(Z, S(n, k, l))$ is path connected. Then $F(X, S(n, k, l))$ is path connected for any compact Hausdorff space X with $\dim X \leq d$.

Corollary

If $\pi_r(S(n, k, l)) = 0$ for each $r \leq d$ then, $F(X, S(n, k, l))$ is path connected for any compact Hausdorff space X with $\dim X \leq d$.

- Fixed $n, k, l \in \mathbb{N}$ with $n > k > l$ find a non vanishing homotopy group of $S(n, k, l)$?

Continuity of path connectedness of $F(X, (S(n, k, l)))$

Theorem

Suppose for each finite simplicial complex Z with $\dim Z \leq d$, the function space $F(Z, S(n, k, l))$ is path connected. Then $F(X, S(n, k, l))$ is path connected for any compact Hausdorff space X with $\dim X \leq d$.

Corollary

If $\pi_r(S(n, k, l)) = 0$ for each $r \leq d$ then, $F(X, S(n, k, l))$ is path connected for any compact Hausdorff space X with $\dim X \leq d$.

- Fixed $n, k, l \in \mathbb{N}$ with $n > k > l$ find a non vanishing homotopy group of $S(n, k, l)$?

Continuity of path connectedness of $F(X,(S(n,k,l)))$

Theorem

Suppose for each finite simplicial complex Z with $\dim Z \leq d$, the function space $F(Z, S(n,k,l))$ is path connected. Then $F(X, S(n,k,l))$ is path connected for any compact Hausdorff space X with $\dim X \leq d$.

Corollary

If $\pi_r(S(n,k,l)) = 0$ for each $r \leq d$ then, $F(X, S(n,k,l))$ is path connected for any compact Hausdorff space X with $\dim X \leq d$.

- Fixed $n,k,l \in \mathbb{N}$ with $n > k > l$ find a non vanishing homotopy group of $S(n,k,l)$?

Thank you!

Some required definitions and results...

Theorem (Chris Phillips)

Let X be a compact, Hausdorff space of dimension d, and let $Y \subset X$ be closed. Let $p, q \in M_n(C(X))$ be projections with the property that $\operatorname{rank}(q(x)) + \left\lfloor \frac{d}{2} \right\rfloor \leq \operatorname{rank}(p(x))$, $\forall x \in X$. Let $s_0 \in M_n(C(Y))$ be such that $s_0^* s_0 = q|_Y$ and $s_0 s_0^* \leq p|_Y$. It follows that there is $s \in M_n(C(X))$ such that $s^* s = q$, $ss^* \leq p$, and $s_0 = s \restriction_Y$.

Corollary (1)

Any trivial projection $q \in M_n(C(Y))$ with $\operatorname{rank}(q) \leq n - \left\lfloor \frac{d}{2} \right\rfloor$, extends to a trivial projection in $M_n(C(X))$.

Some required definitions and results...

Theorem (Chris Phillips)

Let X be a compact, Hausdorff space of dimension d, and let $Y \subset X$ be closed. Let $p, q \in M_n(C(X))$ be projections with the property that $rank(q(x)) + \lfloor \frac{d}{2} \rfloor \leq rank(p(x)), \forall x \in X$. Let $s_0 \in M_n(C(Y))$ be such that $s_0^* s_0 = q|_Y$ and $s_0 s_0^* \leq p|_Y$. It follows that there is $s \in M_n(C(X))$ such that $s^* s = q$, $ss^* \leq p$, and $s_0 = s \upharpoonright_Y$.

Corollary (1)

Any trivial projection $q \in M_n(C(Y))$ with $rank(q) \leq n - \lfloor \frac{d}{2} \rfloor$, extends to a trivial projection in $M_n(C(X))$.

Some required definitions and results...

Corollary (2, Andrew Toms)

Let $q \in M_n(C(X))$ and $F_1, ..., F_k$ be a closed cover of X. $\forall 1 \leq i \leq k$, let $p_i \in Proj(M_n(C(F_i)))$ of constant rank n_i. Assume $n_1 < n_2 < \cdot\cdot < n_k$ and $p_i(x) \leq p_j(x), \forall i \leq j, x \in F_i \cap F_j$. Say $n_i - \text{rank}\,(q) \geq \lfloor \frac{d}{2} \rfloor, \forall 1 \leq i \leq L$. The following hold. If $Y \subset X$ is closed,$q \restriction_Y$ is trivial and,

$$q(y) \leq \bigwedge_{\{i|y \in E_i\}} p_i(y), \forall y \in Y,$$

then $q \restriction_Y$ extends to trivial a projection $\widetilde{q}$ on X with,

$$\widetilde{q}(x) \leq \bigwedge_{\{i|x \in E_i\}} p_i(x), \forall x \in X.$$

Proof of Lemma A

- For each $1 \le i \le L$, let $p_i^{(1)}(x) = p_i(x) - q_1(x), \forall x \in F_i$. Then,

$$p_i^{(1)} \in q_1^{\perp} M_n(C(F_i)) q_1^{\perp} \cong M_{n-n_1+\lfloor \frac{d}{2} \rfloor}(C(F_i))$$

.

- Choose a trivial projection $q_2 \in M_n(F_2)$ with $q_2 \le p_2^{(1)}$ and

$$\text{rank } q_2 = n_2 - (n_1 - \left\lfloor \frac{d}{2} \right\rfloor) - \left\lfloor \frac{d}{2} \right\rfloor = n_2 - n_1 .$$

- Write $X_1 = F_2 \cup F_2 \cup .. \cup F_L$. Extend q_2 to a trivial projection $q_2 \in M_n(C(X_1))$, such that $\forall 2 \le i \le L$,

$$q_2(x) \le p_i^{(1)}(x) , \forall x \in F_i$$

.

- Extend q_2 to a trivial projection $q_2 \in M_n(C(X))$.

Proof of Lemma A

- For each $1 \leq i \leq L$, let $p_i^{(1)}(x) = p_i(x) - q_1(x), \forall x \in F_i$. Then,

$$p_i^{(1)} \in {q_1}^{\perp} M_n(C(F_i)) {q_1}^{\perp} \cong M_{n-n_1+\lfloor \frac{d}{2} \rfloor}(C(F_i))$$

.

- Choose a trivial projection $q_2 \in M_n(F_2)$ with $q_2 \leq p_2^{(1)}$ and

$$\text{rank } q_2 = n_2 - (n_1 - \left\lfloor \frac{d}{2} \right\rfloor) - \left\lfloor \frac{d}{2} \right\rfloor = n_2 - n_1 .$$

- Write $X_1 = F_2 \cup F_2 \cup .. \cup F_L$. Extend q_2 to a trivial projection $q_2 \in M_n(C(X_1))$, such that $\forall 2 \leq i \leq L$,

$$q_2(x) \leq p_i^{(1)}(x) \, , \forall x \in F_i$$

.

- Extend q_2 to a trivial projection $q_2 \in M_n(C(X))$.

Proof of Lemma A

- For each $1 \leq i \leq L$, let $p_i^{(1)}(x) = p_i(x) - q_1(x), \forall x \in F_i$. Then,
$$p_i^{(1)} \in q_1{}^{\perp} M_n(C(F_i)) q_1{}^{\perp} \cong M_{n-n_1+\lfloor \frac{d}{2} \rfloor}(C(F_i))$$
.
- Choose a trivial projection $q_2 \in M_n(F_2)$ with $q_2 \leq p_2{}^{(1)}$ and
$$\text{rank } q_2 = n_2 - (n_1 - \left\lfloor \frac{d}{2} \right\rfloor) - \left\lfloor \frac{d}{2} \right\rfloor = n_2 - n_1 .$$
- Write $X_1 = F_2 \cup F_2 \cup .. \cup F_L$. Extend q_2 to a trivial projection $q_2 \in M_n(C(X_1))$, such that $\forall 2 \leq i \leq L$,
$$q_2(x) \leq p_i{}^{(1)}(x) \, , \forall x \in F_i$$
.
- Extend q_2 to a trivial projection $q_2 \in M_n(C(X))$.

Proof of Lemma A

- For each $1 \leq i \leq L$, let $p_i^{(1)}(x) = p_i(x) - q_1(x), \forall x \in F_i$. Then,

$$p_i^{(1)} \in q_1{}^{\perp} M_n(C(F_i)) q_1{}^{\perp} \cong M_{n-n_1+\lfloor \frac{d}{2} \rfloor}(C(F_i))$$

.

- Choose a trivial projection $q_2 \in M_n(F_2)$ with $q_2 \leq p_2{}^{(1)}$ and

$$\text{rank } q_2 = n_2 - (n_1 - \left\lfloor \frac{d}{2} \right\rfloor) - \left\lfloor \frac{d}{2} \right\rfloor = n_2 - n_1 .$$

- Write $X_1 = F_2 \cup F_2 \cup .. \cup F_L$. Extend q_2 to a trivial projection $q_2 \in M_n(C(X_1))$, such that $\forall 2 \leq i \leq L$,

$$q_2(x) \leq p_i{}^{(1)}(x) \, , \forall x \in F_i$$

.

- Extend q_2 to a trivial projection $q_2 \in M_n(C(X))$.

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\operatorname{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \operatorname{rank}(a(x) + Q_2(x)) &\leq \operatorname{rank} p_1 + \operatorname{rank} q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
  - $\operatorname{rank}(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
  - $\operatorname{rank}(a(x) + Q_L(x)) \leq n_L \leq k, \forall x \in X$.

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \text{rank}(a(x) + Q_2(x)) &\leq \text{rank } p_1 + \text{rank } q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
  - $\text{rank}\,(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
  - $\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k,\ \forall x \in X.$

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \text{rank}(a(x) + Q_2(x)) &\leq \text{rank } p_1 + \text{rank } q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
    - $\text{rank}(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
    - $\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k, \forall x \in X.$

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \text{rank}(a(x) + Q_2(x)) &\leq \text{rank } p_1 + \text{rank } q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
  - rank $(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
  - $\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k, \forall x \in X$.

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \text{rank}(a(x) + Q_2(x)) &\leq \text{rank } p_1 + \text{rank } q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
  - rank $(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
  - $\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k, \forall x \in X$.

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\operatorname{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \operatorname{rank}(a(x) + Q_2(x)) &\leq \operatorname{rank} p_1 + \operatorname{rank} q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
  - rank $(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
  - $\operatorname{rank}(a(x) + Q_L(x)) \leq n_L \leq k$, $\forall x \in X$.

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \text{rank}(a(x) + Q_2(x)) &\leq \text{rank } p_1 + \text{rank } q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
  - rank $(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
  - $\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k$, $\forall x \in X$.

Proof of Lemma A

- Now, $Q_2 = q_1 \oplus q_2 \in M_n(C(X))$ is a trivial projection with

$$\text{rank}(Q_2) = n_1 - \left\lfloor \frac{d}{2} \right\rfloor + (n_2 - n_1) = n_2 - \left\lfloor \frac{d}{2} \right\rfloor$$

- For any $x \in F_2 \cup F_3 \cup .. \cup F_L$, $Q_2(x)(\mathbb{C}^n) \subset a(x)(\mathbb{C}^n)$.
- For any $x \in F_1$,

$$\begin{aligned} \text{rank}(a(x) + Q_2(x)) &\leq \text{rank } p_1 + \text{rank } q_2 \\ &= n_1 + (n_2 - n_1) = n_2 \end{aligned}$$

- Repeat the steps to find a trivial projection $Q_L \in M_n(C(X))$ s.t.,
  - rank $(Q_L) = n_L - \left\lfloor \frac{d}{2} \right\rfloor$
  - $\text{rank}(a(x) + Q_L(x)) \leq n_L \leq k$, $\forall x \in X$.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\text{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$
- Now $p = Q_L + r$ is trivial with rank $p = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$ and,

$$\begin{aligned} \text{rank}(a(x) + p(x)) &\leq \text{rank}(a(x) + Q_L(x)) + \text{rank}(r(x)) \\ &= n_L + (k - n_L) = k \end{aligned}$$

- This completes the proof of Lemma A.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\text{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$
- Now $p = Q_L + r$ is trivial with rank $p = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$ and,

$$\begin{aligned} \text{rank}(a(x) + p(x)) &\leq \text{rank}(a(x) + Q_L(x)) + \text{rank}(r(x)) \\ &= n_L + (k - n_L) = k \end{aligned}$$

- This completes the proof of Lemma A.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\text{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$.
- Now $p = Q_L + r$ is trivial with rank $p = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$ and,

$$\begin{aligned} \text{rank}(a(x) + p(x)) &\leq \text{rank}(a(x) + Q_L(x)) + \text{rank}(r(x)) \\ &= n_L + (k - n_L) = k \end{aligned}$$

- This completes the proof of Lemma A.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\text{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$.
- Now $p = Q_L + r$ is trivial with rank $p = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$ and,

$$\begin{aligned} \text{rank}(a(x) + p(x)) &\leq \text{rank}(a(x) + Q_L(x)) + \text{rank}(r(x)) \\ &= n_L + (k - n_L) = k \end{aligned}$$

- This completes the proof of Lemma A.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\operatorname{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$.
- Now $p = Q_L + r$ is trivial with rank $p = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$ and,

$$\begin{aligned} \operatorname{rank}(a(x) + p(x)) &\leq \operatorname{rank}(a(x) + Q_L(x)) + \operatorname{rank}(r(x)) \\ &= n_L + (k - n_L) = k \end{aligned}$$

- This completes the proof of Lemma A.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \lfloor \frac{d}{2} \rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\text{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$.
- Now $p = Q_L + r$ is trivial with rank $p = k - \lfloor \frac{d}{2} \rfloor \geq l$ and,

$$\begin{aligned} \text{rank}(a(x) + p(x)) &\leq \text{rank}(a(x) + Q_L(x)) + \text{rank}(r(x)) \\ &= n_L + (k - n_L) = k \end{aligned}$$

- This completes the proof of Lemma A.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\mathrm{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$.
- Now $p = Q_L + r$ is trivial with rank $p = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l$ and,

$$\begin{aligned}\mathrm{rank}(a(x) + p(x)) &\leq \mathrm{rank}(a(x) + Q_L(x)) + \mathrm{rank}(r(x)) \\ &= n_L + (k - n_L) = k\end{aligned}$$

- This completes the proof of Lemma A.

Proof of Lemma A

- If $n_L = k$, rank $Q_L = k - \lfloor \frac{d}{2} \rfloor \geq l$. Setting $p = Q_L$ completes the proof of the Lemma.
- If $n_L < k$,

$$\text{rank}(1_n - Q_L) = n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor$$

- Choose a trivial projection $r \in M_n(C(X))$ with,
  - $r(x) \leq (1_n - Q_L)(x), \forall x \in X$
  - rank $r = k - n_L$.
- Now $p = Q_L + r$ is trivial with rank $p = k - \lfloor \frac{d}{2} \rfloor \geq l$ and,

$$\begin{aligned} \text{rank}(a(x) + p(x)) &\leq \text{rank}(a(x) + Q_L(x)) + \text{rank}(r(x)) \\ &= n_L + (k - n_L) = k \end{aligned}$$

- This completes the proof of Lemma A.

Homotopy Classification of Vector bundles

- $V_k(\mathbb{C}^n) = \{(v_1, v_2, ...v_k) \colon v_i \in \mathbb{C}^n$ and $< v_i|v_j >= \delta_{i,j}\}$.
- The complex Grassmann variety, $G_k(\mathbb{C}^n)$, is given by

$$G_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim,$$

  where $(v_1, v_2, ..v_k) \sim (w_1, w_2, ...w_k)$ iff the two k-tuples span the same subspace in $\mathbb{C}^n$.
- We have the natural inclusion, $G_k(\mathbb{C}^n) \subset G_k(\mathbb{C}^{n+1})$, induced by $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. Set $G_k(\mathbb{C}^\infty) = \bigcup_{n\in\mathbb{N}} G_k(\mathbb{C}^n)$.
- Let $\gamma_k^n = (E, \pi, G_k(\mathbb{C}^n))$, where $E = \{(V, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n \colon v \in V\}$, and π is the canonical projection. Then, γ_k^n is a locally trivial vector bundle over $G_k(\mathbb{C}^n)$. Again, $\gamma_k^n \subset \gamma_k^{n+1}$. The resulting direct limit, $\gamma_k = \bigcup_{n\in\mathbb{N}} \gamma_k^n$ determine a vector bundle over $G_k(\mathbb{C}^\infty)$ of dimension k.

Homotopy Classification of Vector bundles

- $V_k(\mathbb{C}^n) = \{(v_1, v_2, ...v_k) \colon v_i \in \mathbb{C}^n \text{ and } < v_i | v_j >= \delta_{i,j}\}$.
- The complex Grassmann variety, $G_k(\mathbb{C}^n)$, is given by

$$G_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim,$$

  where $(v_1, v_2, ..v_k) \sim (w_1, w_2, ...w_k)$ iff the two k-tuples span the same subspace in $\mathbb{C}^n$.
- We have the natural inclusion, $G_k(\mathbb{C}^n) \subset G_k(\mathbb{C}^{n+1})$, induced by $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. Set $G_k(\mathbb{C}^\infty) = \bigcup_{n\in\mathbb{N}} G_k(\mathbb{C}^n)$.
- Let $\gamma_k^n = (E, \pi, G_k(\mathbb{C}^n))$, where $E = \{(V, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n \colon v \in V\}$, and π is the canonical projection. Then, γ_k^n is a locally trivial vector bundle over $G_k(\mathbb{C}^n)$. Again, $\gamma_k^n \subset \gamma_k^{n+1}$. The resulting direct limit, $\gamma_k = \bigcup_{n\in\mathbb{N}} \gamma_k^n$ determine a vector bundle over $G_k(\mathbb{C}^\infty)$ of dimension k.

Homotopy Classification of Vector bundles

- $V_k(\mathbb{C}^n) = \{(v_1, v_2, ...v_k) \colon v_i \in \mathbb{C}^n$ and $< v_i|v_j >= \delta_{i,j}\}$.
- The complex Grassmann variety, $G_k(\mathbb{C}^n)$, is given by

$$G_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim,$$

  where $(v_1, v_2, ..v_k) \sim (w_1, w_2, ...w_k)$ iff the two k-tuples span the same subspace in $\mathbb{C}^n$.
- We have the natural inclusion, $G_k(\mathbb{C}^n) \subset G_k(\mathbb{C}^{n+1})$, induced by $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. Set $G_k(\mathbb{C}^\infty) = \bigcup_{n\in\mathbb{N}} G_k(\mathbb{C}^n)$.
- Let $\gamma_k^n = (E, \pi, G_k(\mathbb{C}^n))$, where $E = \{(V, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n \colon v \in V\}$, and π is the canonical projection. Then, γ_k^n is a locally trivial vector bundle over $G_k(\mathbb{C}^n)$. Again, $\gamma_k^n \subset \gamma_k^{n+1}$. The resulting direct limit, $\gamma_k = \bigcup_{n\in\mathbb{N}} \gamma_k^n$ determine a vector bundle over $G_k(\mathbb{C}^\infty)$ of dimension k.

Homotopy Classification of Vector bundles

- $V_k(\mathbb{C}^n) = \{(v_1, v_2, ...v_k) \colon v_i \in \mathbb{C}^n$ and $< v_i | v_j >= \delta_{i,j}\}$.
- The complex Grassmann variety, $G_k(\mathbb{C}^n)$, is given by

$$G_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim,$$

  where $(v_1, v_2, ..v_k) \sim (w_1, w_2, ...w_k)$ iff the two k-tuples span the same subspace in $\mathbb{C}^n$.
- We have the natural inclusion, $G_k(\mathbb{C}^n) \subset G_k(\mathbb{C}^{n+1})$, induced by $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. Set $G_k(\mathbb{C}^\infty) = \bigcup_{n \in \mathbb{N}} G_k(\mathbb{C}^n)$.
- Let $\gamma_k^n = (E, \pi, G_k(\mathbb{C}^n))$, where $E = \{(V, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n \colon v \in V\}$, and π is the canonical projection. Then, γ_k^n is a locally trivial vector bundle over $G_k(\mathbb{C}^n)$. Again, $\gamma_k^n \subset \gamma_k^{n+1}$. The resulting direct limit, $\gamma_k = \bigcup_{n \in \mathbb{N}} \gamma_k^n$ determine a vector bundle over $G_k(\mathbb{C}^\infty)$ of dimension k.

Homotopy Classification of Vector bundles

- Let $f\colon Y \to X$ be a continuous and $\alpha = (E, p, X)$
- Let $f^*(\alpha) = (E(f^*(\alpha)), \pi, Y)$, with

$$E(f^*(\alpha)) = \{(w, y) \in E \times Y\colon f(y) = p(w)\}$$

and π = restriction of the canonical projection.
- $f^*(\alpha)$ is called the pullback of α to Y via f.

Theorem (Homotopy Classification of Vector bundles)

The function that maps each homotopy class $[f] : X \to G_k(\mathbb{C}^\infty)$ to the isomorphism class of $f^*(\gamma_k)$, is a bijection.

- If X is a finite CW-complex, then $G_k(\mathbb{C}^\infty)$, γ_k of the above theorem can be replaced by $G_k(\mathbb{C}^n)$, γ_k^n, provided $n \geq k + \frac{dim\, X}{2}$

Homotopy Classification of Vector bundles

- Let $f\colon\ Y \to X$ be a continuous and $\alpha = (E, p, X)$
- Let $f^*(\alpha) = (E(f^*(\alpha)), \pi, Y)$, with

$$E(f^*(\alpha)) = \{(w, y) \in E \times Y\colon\ f(y) = p(w)\}$$

  and π = restriction of the canonical projection.
- $f^*(\alpha)$ is called the pullback of α to Y via f.

Theorem (Homotopy Classification of Vector bundles)

The function that maps each homotopy class $[f] : X \to G_k(\mathbb{C}^\infty)$ to the isomorphism class of $f^*(\gamma_k)$, is a bijection.

- If X is a finite CW-complex, then $G_k(\mathbb{C}^\infty)$, γ_k of the above theorem can be replaced by $G_k(\mathbb{C}^n)$, γ_k^n, provided $n \geq k + \frac{\dim X}{2}$

Homotopy Classification of Vector bundles

- Let $f\colon Y \to X$ be a continuous and $\alpha = (E, p, X)$
- Let $f^*(\alpha) = (E(f^*(\alpha)), \pi, Y)$, with

$$E(f^*(\alpha)) = \{(w, y) \in E \times Y\colon f(y) = p(w)\}$$

  and π = restriction of the canonical projection.
- $f^*(\alpha)$ is called the pullback of α to Y via f.

Theorem (Homotopy Classification of Vector bundles)

The function that maps each homotopy class $[f] : X \to G_k(\mathbb{C}^\infty)$ to the isomorphism class of $f^*(\gamma_k)$, is a bijection.

- If X is a finite CW-complex, then $G_k(\mathbb{C}^\infty)$, γ_k of the above theorem can be replaced by $G_k(\mathbb{C}^n)$, γ_k^n, provided $n \geq k + \frac{dim X}{2}$

Homotopy Classification of Vector bundles

- Let $f\colon Y \to X$ be a continuous and $\alpha = (E, p, X)$
- Let $f^*(\alpha) = (E(f^*(\alpha)), \pi, Y)$, with

$$E(f^*(\alpha)) = \{(w, y) \in E \times Y\colon f(y) = p(w)\}$$

  and π = restriction of the canonical projection.
- $f^*(\alpha)$ is called the pullback of α to Y via f.

Theorem (Homotopy Classification of Vector bundles)

The function that maps each homotopy class $[f] : X \to G_k(\mathbb{C}^\infty)$ to the isomorphism class of $f^*(\gamma_k)$, is a bijection.

- If X is a finite CW-complex, then $G_k(\mathbb{C}^\infty)$, γ_k of the above theorem can be replaced by $G_k(\mathbb{C}^n)$, γ_k^n, provided $n \geq k + \frac{\dim X}{2}$

Homotopy Classification of Vector bundles

- Let $f\colon Y \to X$ be a continuous and $\alpha = (E, p, X)$
- Let $f^*(\alpha) = (E(f^*(\alpha)), \pi, Y)$, with

$$E(f^*(\alpha)) = \{(w, y) \in E \times Y\colon f(y) = p(w)\}$$

  and π = restriction of the canonical projection.
- $f^*(\alpha)$ is called the pullback of α to Y via f.

Theorem (Homotopy Classification of Vector bundles)

The function that maps each homotopy class $[f] : X \to G_k(\mathbb{C}^\infty)$ to the isomorphism class of $f^*(\gamma_k)$, is a bijection.

- If X is a finite CW-complex, then $G_k(\mathbb{C}^\infty)$, γ_k of the above theorem can be replaced by $G_k(\mathbb{C}^n)$, γ_k^n, provided $n \geq k + \frac{\dim X}{2}$

Proof of Lemma B

- Union of all $\psi_\alpha^T(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank l, choose α and a projection $p_\alpha \in M_n(C(X))$ with $||p - \psi_\alpha^T(p_\alpha)|| < 1$.
- p is homotopic to $\psi_\alpha^T(p_\alpha)$. Hence $\psi_\alpha^T(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi_\alpha^T(p_\alpha)$ and $q = \psi_\alpha^T(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi_\alpha^T(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As Y_α is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$.

Proof of Lemma B

- Union of all $\psi_\alpha^T(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank l, choose α and a projection $p_\alpha \in M_n(C(X))$ with $||p - \psi_\alpha^T(p_\alpha)|| < 1$.
- p is homotopic to $\psi_\alpha^T(p_\alpha)$. Hence $\psi_\alpha^T(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi_\alpha^T(p_\alpha)$ and $q = \psi_\alpha^T(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi_\alpha^T(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As Y_α is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$.

Proof of Lemma B

- Union of all $\psi_\alpha^T(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank l, choose α and a projection $p_\alpha \in M_n(C(X))$ with $||p - \psi_\alpha^T(p_\alpha)|| < 1$.
- p is homotopic to $\psi_\alpha^T(p_\alpha)$. Hence $\psi_\alpha^T(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi_\alpha^T(p_\alpha)$ and $q = \psi_\alpha^T(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi_\alpha^T(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As Y_α is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$.

Proof of Lemma B

- Union of all $\psi_\alpha^T(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank l, choose α and a projection $p_\alpha \in M_n(C(X))$ with $||p - \psi_\alpha^T(p_\alpha)|| < 1$.
- p is homotopic to $\psi_\alpha^T(p_\alpha)$. Hence $\psi_\alpha^T(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi_\alpha^T(p_\alpha)$ and $q = \psi_\alpha^T(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi_\alpha^T(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As Y_α is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$.

Proof of Lemma B

- Union of all $\psi_\alpha^T(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank l, choose α and a projection $p_\alpha \in M_n(C(X))$ with $||p - \psi_\alpha^T(p_\alpha)|| < 1$.
- p is homotopic to $\psi_\alpha^T(p_\alpha)$. Hence $\psi_\alpha^T(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi_\alpha^T(p_\alpha)$ and $q = \psi_\alpha^T(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi_\alpha^T(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As Y_α is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$.

Proof of Lemma B

- Union of all $\psi_\alpha^T(M_n(C(X_\alpha)))$ is a dense subalgebra of $M_n(C(X))$.
- Fixed a trivial projection $p \in M_n(C(X_\alpha))$ of rank l, choose α and a projection $p_\alpha \in M_n(C(X))$ with $||p - \psi_\alpha^T(p_\alpha)|| < 1$.
- p is homotopic to $\psi_\alpha^T(p_\alpha)$. Hence $\psi_\alpha^T(p_\alpha)$ is also trivial.
- Moreover, this reduces the Lemma to the case $p = \psi_\alpha^T(p_\alpha)$ and $q = \psi_\alpha^T(q_\alpha)$.
- Write $Y_\alpha = \psi_\alpha(X) \subset X_\alpha$. Since $\psi_\alpha^T(p_\alpha)$ is trivial, $p_\alpha|_{Y_\alpha} \in M_n(C(Y_\alpha))$ is trivial.
- As Y_α is closed, by Corollary (1), there is a trivial projection $\tilde{p}_\alpha \in M_n(C(X_\alpha))$ which extends $p_\alpha|_{Y_\alpha}$.

Proof of Lemma B

- May assume, $p = \psi_\alpha^T(\tilde{p}_\alpha)$, $q = \psi_\alpha^T(\tilde{q}_\alpha)$, where $\tilde{p}_\alpha, (\tilde{q}_\alpha) \in M_n(C(X_\alpha))$ are trivial projections of rank l.
- By the homotopy classification of locally trivial vector bundles over CW-complexes, there is a projection valued path H in $M_n(C(X_\alpha))$ connecting $\tilde{p}_\alpha$ and $\tilde{q}_\alpha$.
- Taking $h = H \circ \psi_\alpha$ completes the proof for compact metric spaces.
- For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X. The conclusion now follows from a exact similar argument as before.

Proof of Lemma B

- May assume, $p = \psi_\alpha^T(\tilde{p}_\alpha)$, $q = \psi_\alpha^T(\tilde{q}_\alpha)$, where $\tilde{p}_\alpha, (\tilde{q}_\alpha) \in M_n(C(X_\alpha))$ are trivial projections of rank l.
- By the homotopy classification of locally trivial vector bundles over CW-complexes, there is a projection valued path H in $M_n(C(X_\alpha))$ connecting $\tilde{p}_\alpha$ and $\tilde{q}_\alpha$.
- Taking $h = H \circ \psi_\alpha$ completes the proof for compact metric spaces.
- For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X. The conclusion now follows from a exact similar argument as before.

Proof of Lemma B

- May assume, $p = \psi_\alpha^T(\tilde{p}_\alpha)$, $q = \psi_\alpha^T(\tilde{q}_\alpha)$, where $\tilde{p}_\alpha, (\tilde{q}_\alpha) \in M_n(C(X_\alpha))$ are trivial projections of rank l.
- By the homotopy classification of locally trivial vector bundles over CW-complexes, there is a projection valued path H in $M_n(C(X_\alpha))$ connecting $\tilde{p}_\alpha$ and $\tilde{q}_\alpha$.
- Taking $h = H \circ \psi_\alpha$ completes the proof for compact metric spaces.
- For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X. The conclusion now follows from a exact similar argument as before.

Proof of Lemma B

- May assume, $p = \psi_\alpha^T(\tilde{p}_\alpha)$, $q = \psi_\alpha^T(\tilde{q}_\alpha)$, where $\tilde{p}_\alpha, (\tilde{q}_\alpha) \in M_n(C(X_\alpha))$ are trivial projections of rank l.
- By the homotopy classification of locally trivial vector bundles over CW-complexes, there is a projection valued path H in $M_n(C(X_\alpha))$ connecting $\tilde{p}_\alpha$ and $\tilde{q}_\alpha$.
- Taking $h = H \circ \psi_\alpha$ completes the proof for compact metric spaces.
- For a general compact Hausdorff space X, X is the inverse limit of compact metric spaces each of dimension at most the dimension of X. The conclusion now follows from a exact similar argument as before.