Cartan MASAs and Exact Sequences of Inverse Semigroups

Adam H. Fuller (University of Nebraska - Lincoln)
joint work with Allan P. Donsig and David R. Pitts

COSY 2014, Field's Institute, Toronto

Cartan MASAs

Let $\mathcal{M}$ be a von Neumann algebra. A maximal abelian subalgebra (MASA) $\mathcal{D}$ in $\mathcal{M}$ is a *Cartan MASA* if

- the unitaries $U \in \mathcal{M}$ such that $U\mathcal{D}U^* \subseteq \mathcal{D}$ span a weak-* dense subset in $\mathcal{M}$;
- there is a normal, faithful conditional expectation $E\colon \mathcal{M} \to \mathcal{D}$.

We will call the pair $(\mathcal{M}, \mathcal{D})$ a *Cartan pair*.

What Feldman & Moore did

Feldman and Moore (1977) explored Cartan pairs $(\mathcal{M}, \mathcal{D})$ where $\mathcal{M}_*$ is separable and $\mathcal{D} = L^\infty(X, \mu)$. They showed:

1. there is a measurable equivalence relation R on X with countable equivalence classes and a 2-cocycle σ on R s.t.

   $$\mathcal{M} \simeq \mathbf{M}(R, \sigma) \text{ and } \mathcal{D} \simeq \mathbf{A}(R, \sigma),$$

   where $\mathbf{M}(R, \sigma)$ are "functions on R" and $\mathbf{A}(R, \sigma)$ are the "functions" supported on diag. $\{(x, x) : x \in X\}$;
2. every sep. acting pair $(\mathcal{M}, \mathcal{D})$ arises this way.

What Feldman & Moore did

Feldman and Moore (1977) explored Cartan pairs $(\mathcal{M}, \mathcal{D})$ where $\mathcal{M}_*$ is separable and $\mathcal{D} = L^\infty(X, \mu)$. They showed:

1. there is a measurable equivalence relation R on X with countable equivalence classes and a 2-cocycle σ on R s.t.

$$\mathcal{M} \simeq \mathbf{M}(R, \sigma) \text{ and } \mathcal{D} \simeq \mathbf{A}(R, \sigma),$$

   where $\mathbf{M}(R, \sigma)$ are "functions on R" and $\mathbf{A}(R, \sigma)$ are the "functions" supported on diag. $\{(x, x) : x \in X\}$;
2. every sep. acting pair $(\mathcal{M}, \mathcal{D})$ arises this way.

Feldman and Moore's work is great, and has had a great impact, but has issues:

- Feldman-Moore philosophy is point-based (measure theoretic);
- Feldman-Moore needs equivalence relations with countable equivalence classes and $\mathcal{M}_*$ separable.

Our Objective: Give an alternative approach using algebraic rather than measure theoretic tools which

- conceptually simpler;
- applies to the non-separably acting case.

Inverse Semigroups

A semigroup S is an *inverse semigroup* if for each $s \in S$ there is a unique "inverse" element $s^\dagger$ such that

$$ss^\dagger s = s \text{ and } s^\dagger s s^\dagger = s^\dagger.$$

We denote the idempotents in an inverse semigroup S by $\mathcal{E}(S)$. The idempotents form an abelian semigroup.

Inverse Semigroups

A semigroup S is an *inverse semigroup* if for each $s \in S$ there is a unique "inverse" element $s^\dagger$ such that

$$ss^\dagger s = s \text{ and } s^\dagger s s^\dagger = s^\dagger.$$

We denote the idempotents in an inverse semigroup S by $\mathcal{E}(S)$.
The idempotents form an abelian semigroup.
An inverse semigroup S has a natural partial order defined by

$$s \leq t \text{ if and only if } s = te$$

for some idempotent $e \in \mathcal{E}(S)$.

Extensions of Inverse Semigroups

Let S and P be inverse semigroups. An *idempotent separating extension of S by P* is an inverse semigroup G with

$$P \overset{\iota}{\hookrightarrow} G \overset{q}{\twoheadrightarrow} S$$

and

- ι is an injective homomorphism;
- q is a surjective homomorphism;
- $q(g) \in \mathcal{E}(S)$ if and only if $g = \iota(p)$ for some $p \in P$.

Note that $\mathcal{E}(P) \cong \mathcal{E}(G) \cong \mathcal{E}(S)$.

Extensions of Inverse Semigroups

Definition

Two extensions

$$\begin{array}{ccccc} P & \xrightarrow{\iota_1} & G_1 & \xrightarrow{q_1} & S \\ \| & & & & \| \\ P & \xrightarrow{\iota_2} & G_2 & \xrightarrow{q_2} & S \end{array}$$

of S by P are *equivalent* if there is an isomorphism $\alpha\colon G_1 \to G_2$ such that the diagram commutes

$$\begin{array}{ccccc} P & \xrightarrow{\iota_1} & G_1 & \xrightarrow{q_1} & S \\ \| & & \alpha\downarrow & & \| \\ P & \xrightarrow{\iota_2} & G_2 & \xrightarrow{q_2} & S. \end{array}$$

From Cartan Pairs to Extensions of Inverse Semigroups

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair. Let

$$G = \{v \in \mathcal{M} \text{ a partial isometry} \colon v\mathcal{D}v^* \subseteq \mathcal{D} \text{ and } v^*\mathcal{D}v \subseteq \mathcal{D}\}.$$

From Cartan Pairs to Extensions of Inverse Semigroups

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair. Let

$$G = \{v \in \mathcal{M} \text{ a partial isometry} \colon v\mathcal{D}v^* \subseteq \mathcal{D} \text{ and } v^*\mathcal{D}v \subseteq \mathcal{D}\}.$$

Let

$$P = \{ \text{ partial isometries in } \mathcal{D}\}.$$

From Cartan Pairs to Extensions of Inverse Semigroups

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair. Let

$$G = \{v \in \mathcal{M} \text{ a partial isometry}: v\mathcal{D}v^* \subseteq \mathcal{D} \text{ and } v^*\mathcal{D}v \subseteq \mathcal{D}\}.$$

Let

$$P = \{ \text{ partial isometries in } \mathcal{D}\}.$$

Let μ be the equivalence relation (Munn equivalence) on G given by

$$v \sim w \text{ if and only if } vev^* = wew^* \text{ for all projections} e \in \mathcal{D}.$$

Let $S = G/\mu$. Then S is an inverse semigroup and G is an idempotent separating extension of S by P.

From Cartan Pairs to Extensions of Inverse Semigroups

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair. Let

$$G = \{v \in \mathcal{M} \text{ a partial isometry}\colon v\mathcal{D}v^* \subseteq \mathcal{D} \text{ and } v^*\mathcal{D}v \subseteq \mathcal{D}\}.$$

Let

$$P = \{ \text{ partial isometries in } \mathcal{D}\}.$$

Let μ be the equivalence relation (Munn equivalence) on G given by

$$v \sim w \text{ if and only if } vev^* = wew^* \text{ for all projections} e \in \mathcal{D}.$$

Let $S = G/\mu$. Then S is an inverse semigroup and G is an idempotent separating extension of S by P.

Definition

We call

$$P \hookrightarrow G \to S$$

the *extension associated to* $(\mathcal{M}, \mathcal{D})$.

Uniqueness of Extension

Theorem

Let $(\mathcal{M}_1, \mathcal{D}_1)$ *and* $(M_2, \mathcal{D}_2)$ *be two Cartan pairs with associated extensions*

$$P_i \hookrightarrow G_i \to S_i$$

for $i = 1, 2$.
Suppose there is a normal isomorphism $\theta\colon \mathcal{M}_1 \to \mathcal{M}_2$ *such* $\theta(\mathcal{D}_1) = \mathcal{D}_2$. *Then the two associated extensions are equivalent.*

Uniqueness of Extension

Theorem

Let $(\mathcal{M}_1, \mathcal{D}_1)$ and $(M_2, \mathcal{D}_2)$ be two Cartan pairs with associated extensions

$$P_i \hookrightarrow G_i \to S_i$$

for $i = 1, 2$.
Suppose there is a normal isomorphism $\theta \colon \mathcal{M}_1 \to \mathcal{M}_2$ such $\theta(\mathcal{D}_1) = \mathcal{D}_2$. Then the two associated extensions are equivalent.

It was shown by Laush (1975) that there is one-to-one correspondence between extensions of S by P and the second cohomology group $H^2(S, P)$.

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair, with associated extension $P \to G \to S$. Then S has the following properties

1. S is *fundamental*: $\mathcal{E}(S)$ is maximal abelian in S;
2. $\mathcal{E}(S)$ is a hyperstonean boolean algebra, i.e. the idempotents are the projection lattice of an abelian W^*-algebra;
3. S is a meet semilattice under the natural partial order on S;
4. for every pairwise orthogonal family $\mathcal{F} \subseteq S$, $\bigvee \mathcal{F}$ exists in S.
5. S contains 1 and 0.

Definition

An inverse semigroup S, satisfying the conditions above is called a *Cartan inverse monoid*.

Algebras associated to Cartan Inverse Monoids

From now on, let S be a Cartan inverse monoid.
Let $\widehat{S}$ be the set of characters on S:

$$f\colon S \to \{0,1\} \text{ such that } f(s \wedge t) = f(s)f(t).$$

For each $s \in S$ let

$$G_s := \{f \in \widehat{S} \colon f(s) = 1\}.$$

The sets G_s form a basis for a locally compact topology on $\widehat{S}$.
We are interested in two function algebras:

$$\mathcal{D} := C(\widehat{\mathcal{E}(S)})$$
$$\mathcal{Z} := C_b(\widehat{S})$$

Algebras associated to Cartan Inverse Monoids

From now on, let S be a Cartan inverse monoid.
Let $\widehat{S}$ be the set of characters on S:

$$f \colon S \to \{0,1\} \text{ such that } f(s \wedge t) = f(s)f(t).$$

For each $s \in S$ let

$$G_s := \{f \in \widehat{S} \colon f(s) = 1\}.$$

The sets G_s form a basis for a locally compact topology on $\widehat{S}$.
We are interested in two function algebras:

$$\mathcal{D} := C(\widehat{\mathcal{E}(S)})$$
$$\mathcal{Z} := C_b(\widehat{S})$$

One can describe $\mathcal{D}$ and $\mathcal{Z}$ in a universal manner: $\mathcal{D}$ is the universal algebra for $\wedge$-representations of $\mathcal{E}(S)$;
$\mathcal{Z}$ is the multiplier algebra of the universal algebra for $\wedge$-representations of S.

Extensions of S

Let $\mathcal{D} := C(\widehat{\mathcal{E}(S)})$. Let

$$P := \text{ partial isometries in } \mathcal{D}.$$

Let G be any idempotent-separating extension of S by P.
Goal: Construct a Cartan pair $(\mathcal{M}, \mathcal{D})$ with associated extension $P \hookrightarrow G \to S$.

Representing Extensions

Let G be an idempotent-separating extension of S by P.
As a set,

$$G = \{(s, h) \colon s \in S,\ h \in P,\ s^\dagger s = h^\dagger h\}.$$

We can thus view an element (s, h) in G as function on $\widehat{S}$ with support on $G_{s^\dagger s}$, taking the values of h transposed to have support on G_s.
Thus we view the elements of G as being in $\mathcal{Z} = C_b(\widehat{S})$.

Representing Extensions

Let ψ be a faithful, semifinite weight on $\mathcal{D}$. Then ψ extends to a faithful semifinite weight on $\mathcal{Z}$ such that $\psi(G_{s^\dagger s}) = \psi(G_s)$.

Representing Extensions

Let ψ be a faithful, semifinite weight on $\mathcal{D}$. Then ψ extends to a faithful semifinite weight on $\mathcal{Z}$ such that $\psi(G_{s^\dagger s}) = \psi(G_s)$.
Let (H_ψ, π_ψ) be the GNS-constriction from ψ. There is a dense subset $\mathfrak{n} \subseteq \mathcal{Z}$ and map $\eta\colon \mathfrak{n} \to H_\psi$, with dense image.

Representing Extensions

Let ψ be a faithful, semifinite weight on $\mathcal{D}$. Then ψ extends to a faithful semifinite weight on $\mathcal{Z}$ such that $\psi(G_{s^\dagger s}) = \psi(G_s)$.
Let (H_ψ, π_ψ) be the GNS-constriction from ψ. There is a dense subset $\mathfrak{n} \subseteq \mathcal{Z}$ and map $\eta\colon \mathfrak{n} \to H_\psi$, with dense image.
Define a representation of λ of G on H_ψ by

$$\lambda(v)\eta(w) = \eta(vw).$$

for $v \in G$ and $w \in G \cap \mathfrak{n}$. λ extends to a representation of G on all of H_ψ by partial isometries.

Creating Cartan pairs

Let $\mathcal{M} = \lambda(G)''$, and $\mathcal{D} = \lambda(\mathcal{E}(S))''$. Then $(\mathcal{M}, \mathcal{D})$ is a Cartan pair such that

1. The pair $(\mathcal{M}, \mathcal{D})$ is independent of choice of weight ψ on $\mathcal{D}$;
2. The conditional expectation $E\colon M \to D$ extends the map $E(\lambda((s, h)) = \lambda((s \wedge 1, h|_{s\wedge 1})$.
3. The extension associated to $(\mathcal{M}, \mathcal{D})$ is

$$P \hookrightarrow G \to S$$

(the extension we started with).

Main Theorem

Theorem (Feldman-Moore; Donsig-F-Pitts)

- *If S is a Cartan inverse monoid and $P \hookrightarrow G \xrightarrow{q} S$ is an extension of S by $P := p.i.(C^*(\mathcal{E}(S))$, then the extension determines a Cartan pair $(\mathcal{M}, \mathcal{D})$ which is unique up to isomorphism. Equivalent extensions determine isomorphic Cartan pairs.*
- *Every Cartan pair $(\mathcal{M}, \mathcal{D})$ determines uniquely an extension of a Cartan inverse semigroup S by P, $P \hookrightarrow G \xrightarrow{q} S$.*