Characterization of Spectral Flow

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Outline

- Example
- Definition of spectral flow and context
- Characterization of spectral flow in a type I factor
- Characterization of spectral flow in a type II factor

Disclaimer

No spectres were harmed in the making of this talk.

Example

Hilbert space $\mathcal{H} = L^2(\mathbb{T})$, fix basis $\{h_n := \frac{1}{\sqrt{2\pi}} e^{int}\}_{n\in\mathbb{Z}}$.
Consider $\mathcal{B}(L^2(\mathbb{T}))$.

- self-adjoint unbounded operator $D = \frac{1}{i}\frac{d}{dt}$ (so D maps h_n to $n \cdot h_n$)
- unitary operator u the adjoint of the bilateral shift (maps h_n to h_{n+1}).

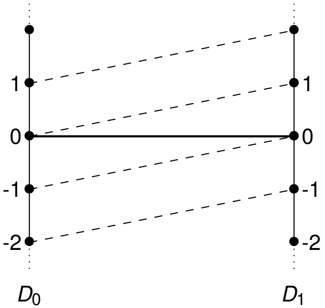
Consider the path $t \mapsto D + t \cdot 1$.

Example (cont'd)

On the previous slide, we defined $D = \frac{1}{i}\frac{d}{dt}$ (so $h_n \mapsto n \cdot h_n$ for $n \in \mathbb{Z}$) and denoted by u the adjoint of the bilateral shift ($h_n \mapsto h_{n+1}$ for $n \in \mathbb{Z}$).

Let $D_0 = D$ and $D_t = D_0 + t \cdot 1$.
Then $D_1 = u^* D u$;
in general, D_t takes h_n to $(n+t)h_n$.

From the picture,
spectral flow($\{D_t\}$) = 1.



spectral flow in $\mathcal{B}(\mathcal{H})$: defined for paths of self-adjoint Fredholm operators (either bounded or unbounded).

Spectral flow in von Neumann algebras: mise-en-scène

$\mathcal{B}(\mathcal{H})$	$\mathcal{N}$ with a semifinite, faithful, normal trace τ
compact operators $\mathcal{K}(\mathcal{H})$	τ-compact operators, $\mathcal{K}_{\mathcal{N}}$ (the norm closed ideal generated by finite trace projections)
Calkin algebra	generalized Calkin algebra ${}^{\mathcal{N}}/_{\mathcal{K}_{\mathcal{N}}}$
Fredholm operators	Breuer-Fredholm operators (operators which are invertible modulo the τ-compacts)

Definitions of Spectral Flow: Bounded Operators

Definition (Phillips, 1997)

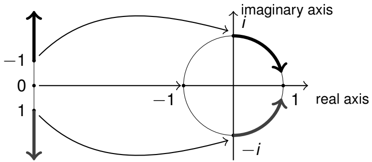
Suppose $\{F_t\}$ is a path of self-adjoint Breuer-Fredholm operators. Let $P_t = \chi_{[0,\infty)}(F_t)$. Then $\pi(P_t)$ is continuous, so we can find indices $i_0, i_1, \ldots i_n$ such that $\|\pi(P_{t_1}) - \pi(P_{t_2})\| < 1$ for all $t_1, t_2 \in [i_k, i_{k+1}]$. This ensures that $P_{t_{i_k}} P_{t_{i_{k+1}}}$ is a Breuer-Fredholm operator when considered as an operator between $P_{t_{i_{k+1}}} \mathcal{H}$ and $P_{t_{i_k}} \mathcal{H}$ and we can define

$$\mathrm{sf}(\{F_t\}) = \sum \mathrm{ind}(P_{t_{i_k}} P_{t_{i_{k+1}}}).$$

Definitions of Spectral Flow: Unbounded Operators

- *gap continuous unbounded operators*

The Cayley map $D \mapsto (D-i)(D+i)^{-1}$ allows us to change a gap-continuous path of unbounded operators to a path of unitary operators.



Definition (Wahl, 2008)

Apply a normalization function Ξ to D_t (warning: $\Xi(D_t)$ is bounded, but $t \mapsto \Xi(D_t)$ is not continuous), and let $U_t = e^{\pi i(\Xi(D_t)+1)}$. Define

$$\mathrm{sf}(\{D_t\}) = \text{winding number}(\{U_t\}) = \frac{1}{2\pi i}\int_0^1 \tau(U_t^{-1}\tfrac{d}{dt}(U_t - 1))\,dt.$$

Context

D - unbounded self-adjoint Breuer-Fredholm operator with $(1+D^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$ and
$u \in \mathcal{N}$ - unitary such that $[D,u]$ is bounded
Let $P = \chi_{[0,\infty]}(D)$ (the projection onto the non-negative spectral subspace of D).

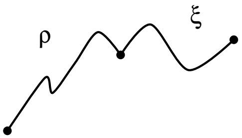
The PuP is a Breuer-Fredholm operator and $\mathrm{ind}(PuP) = \mathrm{sf}(D, uDu^*)$.

This is connected to the pairing between (odd) K-theory and K-homology.

In certain conditions, there are integral formulas for spectral flow. Proving that such a formula calculates spectral flow is a non-trivial proposition, though worth the effort, as having the integral formula allows for different kinds of algebraic manipulation (e.g. the proof of the Local Index Theorem given by Carey, Phillips, Rennie and Sukochev, 2006).

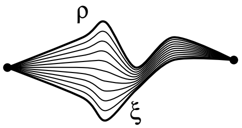
Properties of spectral flow

Concatenation:



$$\mathrm{sf}(\rho * \xi) = \mathrm{sf}(\rho) + \mathrm{sf}(\xi)$$

Homotopy:



$$\mathrm{sf}(\rho) = \mathrm{sf}(\xi)$$

NOTE: can change the homotopy requirement so that ρ and ξ do not have the same endpoints, but the endpoints are invertible operators and remain invertible throughout the homotopy.

Characterization of Spectral Flow (type I_∞ factor)

$\mathcal{CF}_{sa}$ - **unbounded** self-adjoint Fredholm operators (necessarily closed and densely-defined)

Theorem (Lesch, 2005)

Let $\mu : \Omega(\mathcal{CF}_{sa}, (\mathcal{CF}_{sa})^\times) \to \mathbb{Z}$ be a map which satisfies the concatenation and homotopy property (as suggested by the previous slide). Suppose in addition that the following property holds:

'Normalization' property: Fix $T_0 \in (\mathcal{F}_{sa,*})^\times$ with $\sigma(T_0) = \{\pm 1\}$. Suppose that there exists a rank one projection $P \in \mathcal{B}(\mathcal{H})$ such that $(1-P)T_0(1-P) \in \mathcal{B}(P^\perp \mathcal{H})$ is invertible and such that

$$\mu(\{t \oplus P^\perp T_0 P^\perp\}_{t \in [-\frac{1}{2}, \frac{1}{2}]}) = 1.$$

Then $\mu = \mathrm{sf}$.

Overview of proof: Use the gaps in the spectrum to break up the path in such a way that the 'action' is happening on a finite-dimensional corner. Appeal to the result for finite-dimensional matrices to get the result.

Characterization of Spectral Flow (type *II* factor)

Setting: $\mathcal{N}$ is a factor (i.e. the center is trivial)

Theorem

$\mathcal{N}$ - type II factor
$\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^{\times})$ - paths of (bounded) Breuer-Fredholm self-adjoint operators with invertible endpoints
Suppose $\mu : \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^{\times}) \to \mathbb{R}$ is a map which satisfies the following three properties

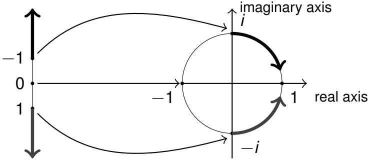
- *homotopy: if $\xi, \rho : \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^{\times})$ and ξ, ρ are homotopic (with endpoints not necessarily fixed, but remaining invertible) then $\mu(\xi) = \mu(\rho)$.*
- *concatenation: if $\xi, \rho \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^{\times})$ with $\rho(1) = \xi(0)$ then $\mu(\rho * \xi) = \mu(\rho) + \mu(\xi)$.*
- *normalization: there exists a finite-trace non-zero projection $P_0 \in \mathcal{N}$ such that if Q, R are projections with $Q \leq P_0$ and $R \leq 1 - Q$ then*

$$\mu(\{ \underbrace{t \oplus 1 \oplus -1}_{\in Q\mathcal{H} \oplus R\mathcal{H} \oplus (Q+R)^{\perp}\mathcal{H}} \}_{t \in [-1,1]}) = \tau(Q).$$

Then μ calculates spectral flow for paths in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^{\times})$.

Cayley map revisited

Recall the Cayley map
$\kappa : D \mapsto (D-i)(D+i)^{-1}$.



Applying the Cayley map to unbounded self-adjoint Breuer-Fredholm operators, we get unitaries U such that $1+U$ is Breuer-Fredholm, and 1 is not an eigenvalue of U.

Denote by
$\mathcal{U}_\kappa$ the unitaries in the image of the Cayley transform (applied to the unbounded self-adjoint Breuer-Fredholm operators), and
$\mathcal{U}_\kappa^{+1}$ the unitaries in $\mathcal{U}_\kappa$ which do not have -1 in the spectrum (ie. corresponding to the self-adjoint invertible operators)

Lemma

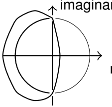
Suppose $\rho \in \Omega(\mathcal{U}_\kappa, {\mathcal{U}_\kappa}^{+1})$ *is such that* $\{-i, i\} \notin \sigma(\rho(t))$ *for any* $t \in [0,1]$. *If* μ *satisfies the concatenation, homotopy and normalization properties then* $\mu(\rho) = \mathrm{sf}(\rho)$.

Lemma

Suppose $\rho \in \Omega(\mathcal{U}_\kappa, {\mathcal{U}_\kappa}^{+1})$ *is such that* $\{-i, i\} \notin \sigma(\rho(t))$ *for any* $t \in [0,1]$*. If* μ *satisfies the concatenation, homotopy and normalization properties then* $\mu(\rho) = \mathrm{sf}(\rho)$*.*

Sketch of proof:

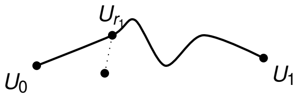
- $P_t = \chi_{[\frac{\pi}{2} \to \frac{3\pi}{2}]}(\rho(t))$ is continuous, which means that $P_t = U_t P_0 U_t^*$ for some path of unitaries $\{U_t\}$



- we can use $\{U_t\}$ to get a homotopy to some path $\{\begin{bmatrix} A_t & 0 \\ 0 & B_t \end{bmatrix}\}$ (with respect to the decomposition $P_0\mathcal{H} \oplus P_0^\perp \mathcal{H}$); moreover, $-1 \notin \sigma(B_t)$.
- construct a second homotopy to $\{\begin{bmatrix} A_t & 0 \\ 0 & B_0 \end{bmatrix}\}$.

Conclude that we must have $\mu(\begin{bmatrix} A_t & 0 \\ 0 & B_0 \end{bmatrix}) = \mathrm{sf}(\begin{bmatrix} A_t & 0 \\ 0 & B_0 \end{bmatrix})$ (using the description of spectral flow for bounded operators in $P_0\mathcal{N}P_0$), and hence $\mu(\rho) = \mathrm{sf}(\rho)$.

Introducing gaps at $\pm i$



On each of the subpaths, can write the operators as $\begin{bmatrix} X_t & V_t \\ W_t & Y_t \end{bmatrix}$ with $-1 \notin \sigma(Y_t)$, and the X_t corner finite-trace. We add the requirement that $\sigma(X_t)$ and $\sigma(X_t - V_t(Y_t + 1)^{-1} W_t)$ should be contained in an arc of length $\frac{\pi}{4}$ around -1. At each division point, we can add little extrusions (as indicated by the dotted line) to get paths with endpoints in $\mathcal{U}_\kappa{}^{+1}$.

A technical lemma now gives us the homotopy which allows us to get a gap in the spectrum at $\pm i$ along each of these new paths.

Technical Lemma

If $U = \begin{bmatrix} X & V \\ W & Y \end{bmatrix}$ (with respect to some decompostion of $\mathcal{H}$) is unitary and $-1 \notin \sigma(Y)$ then, for any fixed $s \in [0,1]$,

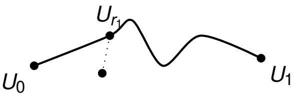
$$Z_s = \begin{bmatrix} X - sV(sY+1)^{-1}W & \sqrt{1-s^2}\,V(sY+1)^{-1} \\ \sqrt{1-s^2}(sY+1)^{-1}W & (Y+s)(sY+1)^{-1} \end{bmatrix}$$

is also unitary. Moreover, the following hold:

- $-1 \notin \sigma(U) \Rightarrow -1 \notin \sigma(Z_s)$.
- if $s \neq 1$ then $1 \notin \sigma(U) \Rightarrow 1 \notin \sigma(Z_s)$.
- if 1 is not an eigenvalue of U then 1 is not an eigenvalue of Z_s *except* in the case when $s = 1$. Note that (for $s = 1$) we have

$$Z_1 = \begin{bmatrix} X - V(Y+1)^{-1}W & 0 \\ 0 & 1 \end{bmatrix}.$$

Introducing gaps at $\pm i$ (cont'd)



We are now dealing with paths in $\Omega(\mathcal{U}_\kappa, {\mathcal{U}_\kappa}^{+1})$ for which we can write the operators as $\begin{bmatrix} X_t & V_t \\ W_t & Y_t \end{bmatrix}$ with $-1 \notin \sigma(Y_t)$, and the X_t corner finite-trace. Moreover, $\sigma(X_t)$ and $\sigma(X_t - V_t(Y_t+1)^{-1}W_t)$ are contained in an arc of length $\frac{\pi}{4}$ around -1.

Apply the magic homotopy indicated by the Technical Lemma at each point along the path simultaneously to get the appropriate holes at $\pm i$ (stop before $s = 1$ in order to ensure 1 is not an eigenvalue).

Conclusion

Given a (gap-continuous) path of self-adjoint Breuer-Fredholm operators, we can homotope it to a path of operators such that the spectrum of each operator has a gap at -i and i. This allows us to reduce the question to the bounded case, and hence conclude that a map which satisfies the homotopy, concatenation and normalization properties must calculate spectral flow.

Thank you!