Homology for one-dimensional solenoids

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Smale spaces

- (X, d) : A compact metric space,
- φ : a homeomorphism of X.

- (X, φ) is a Smale space $\Leftrightarrow$

$$\begin{cases} X^s(x, \varepsilon), & \varepsilon \leq \varepsilon_X \\ X^u(x, \varepsilon), & \varepsilon \leq \varepsilon_X \end{cases}$$

$$d(\varphi(x), \varphi(y)) \leq \lambda d(x, y) \quad \text{on} \quad X^s(x, \varepsilon)$$

$$d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda d(x, y) \quad \text{on} \quad X^u(x, \varepsilon)$$

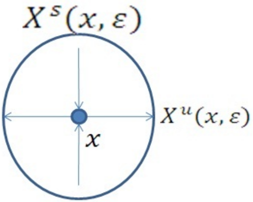


Figure: The local stable and unstable coordinates

Definition

Let (X, φ) and (Y, ψ) be Smale spaces and let $\pi : (Y, \psi) \to (X, \varphi)$ be a map. We say that π is *s*-bijective (or *u*-bijective) if, for any y in Y , its restriction to $Y^s(y, \epsilon)$ (or $Y^u(y, \epsilon)$, respectively) is a local homeomorphic to $X^s(\pi(y), \epsilon)$ (or $X^u(\pi(y), \epsilon)$, respectively).

Examples of Smale spaces:

- The basic sets for Smale's Axiom A systems,
- Substitution tiling spaces,
- Shifts of finite type spaces,
- One-dimensional solenoids.

Shift of finite type spaces

Definition

Let G be a finite (directed)graph:

$$\Sigma_G = \{(e^k)_{k\in\mathbb{Z}} \mid e^k \in G_1 \textit{ and } t(e^k) = i(e^{k+1}), \textit{ for all } k \in \mathbb{Z}\}.$$

The map $\sigma : \Sigma_G \to \Sigma_G$ is the left shift: $\sigma(e)^k = e^{k+1}$, for all $e \in \Sigma_G$.
$(\Sigma_G, \sigma) \Longrightarrow$ is called a shift of finite type space and it is a Smale space with

$$\Sigma^s_G(e, 2^{-k}) = \{f \mid f^i = e^i, i \geq 1 - K\}$$

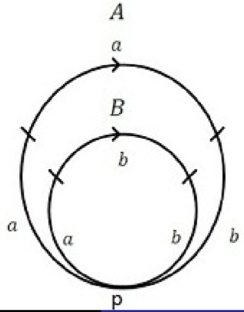
$$\Sigma^u_G(e, 2^{-k}) = \{f \mid f^i = e^i, i \leq k + 1\}$$

One-dimensional solenoids

- Example of one-dimensional solenoid:

X: A wedge of two clockwise circles a, b with a unique vertex p
And

$f : a \to aab, \qquad b \to abb.$



- $$\overline{X} = lim_{\leftarrow} X \leftarrow^{f} X ... = \{(x_0, x_1, x_2, ..) \,|\, f(x_{i+1}) = x_i, i \in \mathbb{N} \cup \{0\}\}$$

- $$\overline{d}((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty}) = \sum_{i=0}^{\infty} 2^{-i} d(x_i, y_i)$$

- $\overline{f}((x_0, x_1, x_2, ..)) = ((f(x_0), f(x_1), f(x_2), ..) = ((fx_0), x_0, x_1, ..)$

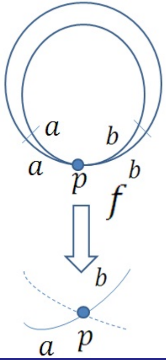
$(\overline{X}, \overline{f})$ is an example of one-dimensional solenoids.

$$\begin{cases} \pi : \overline{X} \to X \\ (x_i)_{i \in \mathbb{N} \cup \{0\}} \to x_0 \end{cases} \Rightarrow \textit{If } x \neq p \Rightarrow \pi^{-1}(x-\epsilon, x+\epsilon) \approx (x-\epsilon, x+\epsilon) \times \textit{Sequence space}$$

How about point p:

$$f(U_p) \approx (-1,1) \text{ (The flatting condition)}$$

$$\pi^{-1}(U_p) \approx (-1,1) \times \text{sequence set}$$



Definition

[Williams, Yi, Thomsen]Let X be a finite (unoriented), connected graph with vertices V and edges E. Consider a continuous map $f : X \to X$. We say that (X, f) is a pre-solenoid if the following conditions are satisfied for some metric d giving the topology of X:

α) (expansion) there are constants $C > 0$ and $\lambda > 1$ such that $d(f^n(x), f^n(y)) \geq C\lambda^n d(x, y)$ for every $n \in \mathbb{N}$ when $x, y \in e \in E$ and there is an edge $e' \in E$ with $f^n([x, y]) \subset e'$ ($[x, y]$ is the interval in e between x and y),

β) (non-folding) f^n is locally injective on e for each $e \in E$ and each $n \in \mathbb{N}$,

γ) (Markov) $f(V) \subset V$, for every edge $e \in E$,

δ) (mixing) there is $m \in \mathbb{N}$ such that $X \subseteq f^m(e)$,for each $e \in E$.

ϵ) (flattening) there is $l \in \mathbb{N}$ such that for all $x \in X$ there is a neighbourhood U_x of x with $f^l(U_x)$ homeomorphic to $(-1, 1)$.

Suppose that (X, f) is a pre-solenoid:

$$\overline{X} = \{(x_i)_{i=0}^{\infty} \in X^{\mathbb{N}\cup\{0\}} : f(x_{i+1}) = x_i, i = 0, 1, 2, \cdots\}$$

Then X is a compact metric space with the metric:

$$\overline{d}((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty}) = \sum_{i=0}^{\infty} 2^{-i} d(x_i, y_i).$$

We also define $\overline{f} : \overline{X} \to \overline{X}$ by

$$\overline{f}(x)_i = f(x_i)$$

Definition

Let (X, f) be a pre-solenoid. The system $(\overline{X}, \overline{f})$ is called a generalized one-dimensional solenoid.

Theorem

[Thomsen]One-dimensional generalized solenoids are Smale spaces whose $X^u(x, \epsilon)$ is homeomorphism to $(-1, 1)$ and $X^s(x, \epsilon)$ is disconnected set for every $x \in \overline{X}$

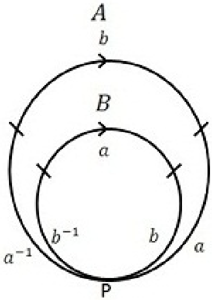
Theorem

[Williams] Let $(\overline{X}, \overline{f})$ be a 1-solenoid. Then there is an integer n and pre-solenoid (X', f') such that $(\overline{X}, \overline{f^n})$ is conjugate to $(\overline{X'}, \overline{f'})$ and X' has a single vertex That is, X' is a wedge of circles.

$$One-dimensional Solenoids : \begin{cases} Orientable, \\ Unorientable. \end{cases}$$

X: A wedge of two clockwise circles a, b with a unique vertex p
And

$$g : a \to a^{-1}ba, b \to b^{-1}ab.$$



$\Rightarrow (X, g)$ represents an unorientable one-dimensional solenoids.

An s/u-bijective pair $(Y, \psi, \pi_s, Z, \zeta, \pi_u)$:

$\pi_s : (Y, \psi) \to (X, \varphi)$ is $s-$bijective map and $Y^u(y, \epsilon)$ is totally disconnected set,

$\pi_u : (Z, \zeta) \to (X, \varphi)$ is $u-$bijective map and $Z^s(z, \epsilon)$ is totally disconnected set,

For $(\overline{X}, \overline{f})$:

$(Y, \psi) = ?, \quad \pi_s = ?$ and $(Z, \zeta) = (\overline{X}, \overline{f}), \quad \pi_u = I_{\overline{X}}$

Lemma (Yi)

Suppose that (X, f) *is a pre-solenoid with a single vertex* p*. Let* $E = \{e_1, ...e_m\}$ *be the edge set of* X *with a given orientation. For each edge* $e_i \in E$*, we can give* $e_i - f^{-1}\{p\}$ *the partition* $\{e_{i,j}\}, 1 \leq j \leq j(i)$ *such that* $f(e_{i,j}) \in E$*.*

According to this partition, we define a gragh G:

$$G : \begin{cases} G^0, & \textit{The edges of } X \\ G^1, & e_i \to e_j \Leftrightarrow f(e_{il}) = e_j. \end{cases}$$

Theorem (Yi)

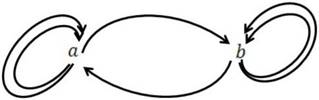
Suppose $(\overline{X}, \overline{f})$ *is one-dimensional solenoids. Then there is a factor map* $\rho : (\Sigma_G, \sigma) \to (\overline{X}, \overline{f})$ *such that* ρ *is s-bijective and at most two to one.*

Theorem

$(\Sigma_G, \sigma, \rho, \overline{X}, \overline{f}, I_{\overline{X}})$ is an s/u-bijective pair for each one-dimensional solenoids.

- According to the flatting Axiom, there are two edges e_1, e_2 such that $f(U_p) \subset e_1 \cup e_2$.
- $w = \sum_{f(e_{i1})=f(e_{ij(i)})=e_1} e_i - \sum_{f(e_{i1})=f(e_{ij(i)})=e_2} e_i \in \mathbb{Z}G^1$

(X, f) : $f : a \to aabb \to abb$
(X, g) : $f : a \to, \ a^{-1}ba \to bb^{-1}ab$



$\Rightarrow$ But $(X, f) \Rightarrow w = 0, \quad (X, g) \Rightarrow w = a - b \neq 0$

Theorem

Let (X, f) be a pre-solenoid. Then $w = 0$ if and only if $(\overline{X}, \overline{f})$ is orientable.

One-dimensional solenoids

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & D(\Sigma_{0,0}) & \to & D(\Sigma_{0,1}) & \to & D(\Sigma_{0,2}) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & D(\Sigma_{1,0}) & \to & D(\Sigma_{1,1}) & \to & D(\Sigma_{1,2}) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & D(\Sigma_{2,0}) & \to & D(\Sigma_{2,1}) & \to & D(\Sigma_{2,2}) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Theorem

Let (X, f) be a pre-solenoid and $(\overline{X}, \overline{f})$ be its associated one-solenoid. If (X, f) is orientable, then

$$H^s_N(\overline{X}, \overline{f}) = \begin{cases} D^s(\Sigma_X, \sigma) & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

If (X, f) is not orientable, then

$$H^s_N(\overline{X}, \overline{f}) = \begin{cases} D^s(\Sigma_X, \sigma)/< 2[w, 1] > & N = 0, \\ 0 & N \neq 0. \end{cases}$$

Theorem

Let (X, f) be a pre-solenoid and $(\overline{X}, \overline{f})$ be its associated one-solenoid. If (X, f) is orientable, then

$$H^u_N(\overline{X}, \overline{f}) = \begin{cases} D^u(\Sigma_X, \sigma) & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

If (X, f) is not orientable, then

$$H^u_N(\overline{X}, \overline{f}) = \begin{cases} Ker(w^*) & N = 0, \\ \mathbb{Z}_2 & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

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THANK YOU