Recurrence and Orbit Equivalence

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A work under progress with

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Field's Institute, June 2014

Dynamical Systems

$(X,\ T)$ is a Cantor Minimal System. $\mathcal{M}_T(X) = \{\mu : \ T\mu = \mu\}$.

- Weakly Mixing: (TFAE)
  $(X \times X,\ T \times T)$ is transitive.

  $\forall$ two open sets U, V $\{n \in \mathbb{N} : \ U \cap T^n V \neq \emptyset\}$ is thick.

- Spectrum:

  $\lambda = e^{2\pi i\theta} \in SP(T)$ if $\exists\ f_\lambda \in C(X);\ \ f_\lambda \circ T = \lambda f_\lambda$.

- factoring into (onto) the unit circle:

$$f_\lambda \circ T = \lambda f_\lambda, \qquad \begin{array}{ccc} X & \xrightarrow{T} & X \\ f_\lambda \downarrow & & \downarrow f_\lambda \\ S^1 & \xrightarrow{R_\theta} & S^1 \end{array}$$

Weak Mixing $\Leftrightarrow$ trivial spectrum, $\theta = 0$.

- Kronecker Sys.: A minimal equicontinuous system on a compact metric group. It is also called an automorphic system.
- An almost 1:1 extension of a Kronecker system is called almost automorphic.

- Λ : the set of all rationally independent elements of $SP(T)$.

$$S : \prod_{\lambda \in \Lambda} S^1 \to \prod_{\lambda \in \Lambda} S^1$$

$$S((x_1, x_2, \dots)) = (\lambda_1, \lambda_2, \dots)(x_1, x_2, \dots).$$

- (Z_r, T_r) the maximal rational factor (an odometer) of (X, T),

$$Z = \prod_{\theta} S^1_\theta \times Z_r, \qquad R = \prod_{\theta} R_\theta \times T_r \qquad \begin{array}{ccc} X & \xrightarrow{T} & X \\ \phi \downarrow & & \downarrow \phi \\ Z & \xrightarrow{R} & Z \end{array}$$

(Z, R) is the maximal equicontinuous factor of (X, T).

Weak Mixing $\Leftrightarrow$ trivial maximal equicontinous factor

(Strong) Orbit Equivalence

- $(X,\ T)$ O.E. $(Y,\ S)$:

$$\exists\ h : X \to Y; \quad h(\mathcal{O}_T(x)) = \mathcal{O}_S(h(x))$$

- cocycle map: $\quad n : X \to \mathbb{Z} \quad\quad h(Tx) = S^{n(x)}x$ is unique.

- $(X,\ T)$ S.O.E. $(Y,\ S)$:
  the cocycle map has just one point of discontinuity.

- $[\mathrm{Giordano, Putnam, Skau,' 95}]$
  A uniquely ergodic CMS is O.E. to an Odometer or a Denjoy's.

So there are two types of orbit equivalence classes that they are both related to an almost automorphic system.

Questions:

- What ican we say for Strong Orbit Equivalence?
- For non-uniquely ergodic systems, what can we say for different types (in terms of recurrence) of dynamics in one Orbit equivalence class?
- Is it true that in any strong orbit equivalence class, there exists an almost automorphic system?

Strategy

- $$\mathrm{CMS}\begin{cases} \text{1) infinite spectrum,} \\ \\ \text{2) finite spectrum and non-weakly mixing} \\ \\ \text{3) weakly mixing} \end{cases}$$

The second one will be turned into the other ones. Indeed,

$$SP(T) = \{e^{2\pi i \frac{1}{q}}, \cdots, e^{2\pi i \frac{q-1}{q}}\} \quad \Rightarrow \quad X = X_1 \cup \cdots \cup X_q$$

such that $T^q|_{X_i}$, $1 = 1, 2, \cdots, q$ is minimal and it is weakly mixing or having irrational spectrum.

We observe that

- infinite set of spectrum:

  Let (X, T) be a Cantor minimal system with infinite spectrum described by (Z, R). Then there exists a minimal almost 1:1 Extension (almost automorhic), (Y, S), such that (Y, S) is (strong) orbit equivalent to (X, T).

$$\begin{tikzcd}
(Y, S) \arrow[rr, phantom, "(S.)O.E."] \arrow[dr] & & (X, T) \arrow[dl] \\
& (Z, R) &
\end{tikzcd}$$

- weakly mixing

  For any Denjoy's there exists some weakly mixing systems in its strong orbit equivalence class. But there are some counterexamples for the second and the third questions.

Dimension groups:

- $D(X,\ T) = C(X,\mathbb{Z})/\{f - f\circ T:\ f\in C(X,\mathbb{Z})\}$,
- $D_m(X,\ T) = C(X,\ T)/\{f:\ \int f\ d\mu = 0,\ \forall\mu\in\mathcal{M}_T(X)\}$,

  Remark. $D_m(X,\ T) = D(X,\ T)/Inf(D(X,\ T))$.

- Theorem.[GPS,'95] $(X,\ T)$ O.E. $(Y,\ S)$ iff

  $(D_m(X,\ T), D_m^+(X,\ T), [1_X]) \simeq (D_m(Y,\ T), D_m^+(Y,\ T), [1_Y])$.

- Theorem.[GPS,'95] $(X,\ T)$ S.O.E. $(Y,\ S)$ iff

  $(D(X,\ T), D^+(X,\ T), [1_X]) \simeq (D(Y,\ T), D^+(Y,\ T), [1_Y])$.

- Real coboundaries: $(1-T)C(X,\mathbb{R})$
- Theorem. Let (X,T) be a CMS. $\lambda = \exp(2\pi i\theta)$, $0<\theta<1$ is an eigenvalue, iff

$$\exists \text{ clopen } U_\lambda; \quad 1_{U_\lambda} - \theta 1_X = F - F\circ T \in (1-T)C(X,\mathbb{R}).$$

- Corollary. Let (X,T) be a CMS. $\lambda = \exp(2\pi i\theta)$, $0<\theta<1$ is an eigenvalue, then

$$\exists \text{ clopen } U; \quad \mu(U) = \theta$$

$\forall\, \mu \in \mathcal{M}_T(X)$.

- Theorem. Let $(X,\ T)$ be a CMS and $(Z,\ R)$ be the CMS which describes the $SP(T)$. Then

$$D_m(Z,\ R) =< \{[1_{U_\lambda}] : \quad \lambda \in SP(T)\} > .$$

- Remark. If for any finite number of irrational eigenvalues, say $\{e^{2\pi i\theta_j}\}_{j=1}^n$, the set $\{1, \prod_{j=1}^n \theta_j\}$ are rationally independent, then

$$D(Z,\ R) =< \{[1_{U_\lambda}] : \quad \lambda \in SP(T)\} > .$$

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$$\begin{gathered} \iota : D_m(Z,\ R) \to D_m(X,\ T) \\ [1_{U_\lambda}] \mapsto [1_{V_\lambda}] \end{gathered}$$

- Theorem. Let $(X,\ T)$ and $(Z,\ R)$ be as before. Then

$$D_m(X,\ T)/D_m(Z,\ R)$$

is torsion free. In particular,

$$0 \to D_m(Z, R) \to D_m(X,\ T) \to D_m(X,\ T)/D_m(Z,\ R) \to 0.$$

is a short exact sequence.

- Theorem. [Sugisaki, 2011] Let (Z, R) be a uniquely ergodic CMS and G be a simple dimension group such that

$$0 \to D_m(Z,R) \to G \to G/D_m(Z,\, R) \to 0$$

is a short exact sequence. Then

$$\exists \text{ an almot } 1:1 \text{ extension}, (Y,\, S), \text{ with } \; D_m(Y,\, S) = G.$$

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$$\begin{array}{ccccc} (Y,\, S) & & (S.)O.E. & & (X,\, T) \\ & \searrow & & \swarrow & \\ & & (Z,\, R) & & \end{array}$$

Now we are going to examine the strong orbit equivalence class of weakly mixing systems in having an almost automorphic system.

- Theorem. [Ormes, '97] Let (X, T) be a CMS and (Y, S, μ) be an ergodic system that the rational spectrum of T is contained in the measurable spectrum of S. Then there exists a CMS, (X, T'), with an ergodic invariant measure ν on X such that

$$(Y, S, \mu) \simeq (X, T', \nu) \qquad \text{and} \qquad (X, T) \text{ S.O.E. } (X, T')$$

- Using Bratteli diagram and combinatorial properties of weak mixing:

  Theorem. If (X, T) is of "finite rank" and has trivial rational spectrum then, there exists a topologically weakly mixing system, (X, S), strongly orbit equivalent to it.

- Spectrum $\begin{cases} \text{i) rational numbers in the spectrum} \\ \\ \text{ii) irrational numbers in the spectrum} \end{cases}$
- i) [GPS, '95] the rational subgroup of $D(X,\ T)$ is not trivial.

$$Q(D(X,\ T)) = \{[g] :\ \exists n, k \in \mathbb{Z},\quad n[g] = k[1_X]\} \neq \mathbb{Z}.$$

- ii) the set of values of the traces contains irrational numbers.

$$\exists\ \theta \in \mathbb{Q}^c,\ \exists\ U \text{ clopen};\ \mu(U) = \theta,\ \forall \mu \in \mathcal{M}_T(X).$$

$$\tau(D(X,\ T)) \cap \mathbb{Q}^c \cap [0,1] = \{\mu(U) :\quad U \text{ is clopen}\} \cap \mathbb{Q}^c \neq \emptyset.$$

Example:

- 1) Let $\sigma : \{0,1\} \to \{0,1\}^+$ be the substitution with

$$\sigma(0) = 001, \qquad \sigma(1) = 00111.$$

Let $(B,\ V)$ be the Stationary Bratteli diagram associated to it with incidence matrix:

$$M_n = M = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \xrightarrow{\tau} \tfrac{1}{2}\mathbb{Z}[\tfrac{1}{2}] \longrightarrow 0$$

- $[1_X] = (1,\ 1) \Rightarrow Q(D(X,\ T)) \simeq \mathbb{Z}$

  $\tau(D(X,\ T)) \subset \mathbb{Q}$
- can not be strongly orbit equivalent to an almost automorphic system.