Exact C*-algebras and $C_0(X)$-structure

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Canadian Annual Symposium on Operator Algebras and Their Applications
June 2014

$C_0(X)$-algebras

Definition (Kasparov)

Let X be a locally compact Hausdorff space and A a C*-algebra. If there exists a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ with the property that $\mu_A(C_0(X)) \cdot A$ is dense in A, we say that the triple (A, X, μ_A) is a $C_0(X)$-*algebra* .

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For $x \in X$, define

- $C_{0,x}(X) = \{f \in C_0(X) : f(x) = 0\}$, and note that $C_{0,x}(X) \cdot A$ is a closed two-sided ideal of A,

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- $C_{0,x}(X) = \{f \in C_0(X) : f(x) = 0\}$, and note that $C_{0,x}(X) \cdot A$ is a closed two-sided ideal of A,
- $A_x = \dfrac{A}{C_{0,x}(X) \cdot A}$ the quotient C*-algebra, and
- $\pi_x : A \to A_x$ the quotient homomorphism.

$C_0(X)$-algebras and C*-bundles

We regard A as an algebra of sections of $\coprod_{x \in X} A_x$, identifying each $a \in A$ with $\hat{a} : X \to \coprod_{x \in X} A_x$, where

$$\hat{a}(x) = \pi_x(a)$$

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For all $a \in A$, we have

- $\|a\| = \sup_{x\in X} \|\pi_x(a)\|$,
- the function $X \to \mathbb{R}_+$, $x \mapsto \|\pi_x(a)\|$ is upper-semicontinuous, and vanishes at infinity on X.

Thus, we think of a $C_0(X)$-algebra as the algebra of sections (vanishing at infinity) of a C*-bundle over X.

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Thus, we think of a $C_0(X)$-algebra as the algebra of sections (vanishing at infinity) of a C*-bundle over X.
If for all $a \in A$, the norm functions $x \mapsto \|\pi_x(a)\|$ are continuous on X, then we say that (A, X, μ_A) is a *continuous $C_0(X)$-algebra*.

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Interest in $C_0(X)$-algebras and C*-bundles: to decompose the study of a given C*-algebra A into that of

- the fibre algebras A_x,
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Question: For two C*-algebras A and B, let $A \otimes B$ denote their minimal tensor product. Given a $C_0(X)$-algebra structure on A and a $C_0(Y)$-algebra structure on B, what can be said about $A \otimes B$ as a $C_0(X \times Y)$-algebra?

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Related question: ideal structure of $A \otimes B$?

Ideals of $A \otimes B$

If $I \lhd A$ and $J \lhd B$, let $q_I : A \to A/I$ and $q_J : B \to B/J$ be the quotient maps. Then $q_I \odot q_J : A \odot B \to (A/I) \odot (B/J)$ has

$$\ker(q_I \odot q_J) = I \odot B + A \odot J,$$

which, by injectivity, has closure

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Extending $q_I \odot q_J$ to $q_I \otimes q_J : A \otimes B \to (A/I) \otimes (B/J)$ gives a closed two-sided ideal

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Clearly

$$\ker(q_I \otimes q_J) \supseteq I \otimes B + A \otimes J.$$

but this inclusion may be strict.

The fibrewise tensor product

- Let (A, X, μ_A) be a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, and denote by $\pi_x : A \to A_x$ and $\sigma_y : B \to B_y$ the quotient $*$-homomorphisms, where $x \in X, y \in Y$.

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- We get $*$-homomorphisms $\pi_x \otimes \sigma_y : A \otimes B \to A_x \otimes B_y$,
- Hence we may regard $A \otimes B$ as an algebra of sections of $\coprod\{A_x \otimes B_y : (x, y) \in X \times Y\}$, where $c \in A \otimes B$ is identified with

$$\begin{array}{rcl} \hat{c} : X \times Y & \to & \coprod\{A_x \otimes B_y : (x, y) \in X \times Y\} \\ \hat{c}((x, y)) & = & (\pi_x \otimes \sigma_y)(c). \end{array}$$

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- This construction gives a C*-bundle decomposition of $A \otimes B$ (Kirchberg & Wassermann).

$A \otimes B$ as a $C_0(X \times Y)$-algebra

Since $C_0(X) \otimes C_0(Y) \equiv C_0(X \times Y)$ and $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes B)$, we get a $*$-homomorphism

$$\mu_A \otimes \mu_B : C_0(X \times Y) = C_0(X) \otimes C_0(Y) \to ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes B).$$

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For $(x, y) \in X \times Y$, it can be shown that

$$C_{0,(x,y)}(X \times Y) \cdot (A \otimes B) = (C_{0,x}(X) \cdot A) \otimes B + A \otimes (C_{0,y}(Y) \cdot B)$$

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$$\begin{aligned} (A \otimes B)_{(x,y)} &= \frac{A \otimes B}{\ker(\pi_x) \otimes B + A \otimes \ker(\sigma_y)} \\ &\neq A_x \otimes B_y, \end{aligned}$$

in general.

Continuity of the fibrewise tensor product

Clearly we have

$$\begin{aligned} &(A \otimes B)_{(x,y)} \equiv A_x \otimes B_y \\ &\Leftrightarrow \ker(\pi_x) \otimes B + A \otimes \ker(\sigma_y) = \ker(\pi_x \otimes \sigma_y). \end{aligned} \tag{$F_{X,Y}$}$$

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Theorem (Kirchberg & Wassermann)

Let (A, X, μ_A) be a continuous $C_0(X)$-algebra and (B, Y, μ_B) a continuous $C_0(Y)$-algebra. Then the norm functions

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are continuous on $X \times Y$ for all $c \in A \otimes B$ if and only if $(F_{X,Y})$ holds.

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Note that if $(F_{X,Y})$ holds, then this also implies that the $C_0(X \times Y)$-algebra $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.

Continuity of the $C_0(X \times Y)$-algebra $A \otimes B$

By contrast, we have shown that there exist (A, X, μ_A) and (B, Y, μ_B), both continuous, such that

- the fibrewise tensor product of (A, X, μ_A) and (B, Y, μ_B) is discontinuous, but
- the $C_0(X \times Y)$-algebra $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.

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- Let $A = \prod_{n \geq 1} M_n(\mathbb{C})$, then A defines a continuous $C(\beta\mathbb{N})$-algebra, with fibres $A_n = M_n(\mathbb{C})$ for $n \in \mathbb{N}$.

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- Then $(A \otimes B, \beta\mathbb{N}, \mu_A \otimes 1)$ is a continuous $C(\beta\mathbb{N})$-algebra,

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- Then $(A \otimes B, \beta\mathbb{N}, \mu_A \otimes 1)$ is a continuous $C(\beta\mathbb{N})$-algebra, but there is $p \in \beta\mathbb{N} \backslash \mathbb{N}$ such that
  - $(A \otimes B)_p \neq A_p \otimes B$ (i.e. property $(F_{X,Y})$ fails) and
  - $p \mapsto \|(\pi_p \otimes \mathrm{id})(c)\|$ is discontinuous at p for some $c \in A \otimes B$, hence the fibrewise tensor product is a discontinuous C*-bundle.

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  - $p \mapsto \|(\pi_p \otimes \mathrm{id})(c)\|$ is discontinuous at p for some $c \in A \otimes B$, hence the fibrewise tensor product is a discontinuous C*-bundle.
- In fact this occurs whenever B is an inexact C*-algebra.

Property (F)

- Let A and B be C*-algebras. If for all ideals $I \lhd A$ and $J \lhd B$ we have

$$\ker(q_I \otimes q_J) = I \otimes B + A \otimes J, \tag{F}$$

then $A \otimes B$ is said to satisfy Tomiyama's property (F).

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- Given (A, X, μ_A) and (B, Y, μ_B), $(F) \Rightarrow (F_{X,Y})$, which in turn implies
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- A is *exact* iff $A \otimes B$ satisfies (F) for all B.

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  - $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ has fibres $A_x \otimes B_y$, and
  - if A and B are continuous then so is $A \otimes B$.
- A is *exact* iff $A \otimes B$ satisfies (F) for all B.
- If (A, X, μ_A) is continuous and A exact, then $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous whenever (B, Y, μ_B) continuous.

Continuity and exactness

Theorem (M.)

Let (A, X, μ_A) be a continuous $C_0(X)$-algebra. TTFAE:

(i) *A is exact,*

(ii) *for every continuous $C_0(Y)$-algebra B, the $C_0(X \times Y)$-algebra $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.*

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Analogous result for the fibrewise tensor product due to Kirchberg and Wassermann: A exact $\Leftrightarrow$ fibrewise tensor product continuous for all B.

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Analogous result for the fibrewise tensor product due to Kirchberg and Wassermann: A exact $\Leftrightarrow$ fibrewise tensor product continuous for all B.

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Continuity and exactness

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By contrast, given continuous (A, X, μ_A) and (B, Y, μ_B), we have

- fibrewise tensor product of A and B continuous $\Leftrightarrow$ $(F_{X,Y})$ holds,
- $(F_{X,Y})$ holds $\Rightarrow$ $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ continuous, but
- $(A \otimes B, X \times Y, \mu_A \otimes \mu_B)$ continuous $\not\Rightarrow$ $(F_{X,Y})$ holds.

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Let A be a C*-algebra, and $\hat{Z}$ the maximal (primitive) ideal space of $ZM(A)$, so that we have an isomorphism $\theta_A : C(\hat{Z}) \equiv ZM(A)$.

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Remark: $\mathrm{Glimm}(A)$ may be constructed from the topological space $\mathrm{Prim}(A)$ of primitive ideals of A (with the hull kernel topology) alone; no need for multiplier algebras.

Characterisation of $\mathrm{Glimm}(A \otimes B)$

Theorem (M.)

Let A and B be C^*-algebras. Then the map

$$\begin{aligned} \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) &\to \mathrm{Glimm}(A \otimes B) \\ (G_p, G_q) &\mapsto G_p \otimes B + A \otimes G_q \end{aligned}$$

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Remarks:

1. In the case that $A \otimes B$ satisfies property (F), the 'open bijection' part was shown by Kaniuth.
2. In general, the topology on $\mathrm{Glimm}(A \otimes B)$ depends only on the product space $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$.

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(iii) *For every C^*-algebra B and $(p, q) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, we have*

$$A \otimes G_q + G_p \otimes B = \ker(\pi_p \otimes \sigma_q),$$

with $\pi_p : A \to A/G_p, \sigma_q : B \to B/G_q$ the quotient maps.

$C_0(\mathrm{Glimm}(A))$-representations

If $\mathrm{Prim}(A)$ is Hausdorff in the hull-kernel topology, then

- $\mathrm{Prim}(A) = \mathrm{Glimm}(A)$ as sets of ideals and topologically, and
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Definition (Archbold & Somerset)

A separable C*-algebra A is called *quasi-standard* if

- $(A, \mathrm{Glimm}(A), \theta_A)$ is a continuous $C_0(\mathrm{Glimm}(A))$-algebra, and
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Examples of quasi-standard C*-algebras: all von Neumann algebras, local multiplier algebras, and many group C*-algebras.

Classes of Dauns-Hofmann representations

We have the following relations:

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Question: are these classes closed under tensor products?

Characterisation of exactness

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Let A be a C^*-algebra.

(i) *If* $(A, \mathrm{Glimm}(A), \theta_A)$ *is a continuous* $C_0(\mathrm{Glimm}(A))$*-algebra, then* A *is exact* $\Leftrightarrow$ *for all* C^**-algebras* B *with* $(B, \mathrm{Glimm}(B), \theta_B)$ *continuous, the* $C_0(\mathrm{Glimm}(A \otimes B))$*-algebra* $(A \otimes B, \mathrm{Glimm}(A \otimes B), \theta_A \otimes \theta_B)$ *is continuous,*

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Theorem

Let A be a unital quasi-standard C-algebra. Then A is nuclear $\Leftrightarrow$ $A \otimes_{\max} B$ is quasi-standard for all quasi-standard C*-algebras B.*