Noncommutative Geometry and Conformal Geometry
(joint work with Hang Wang)

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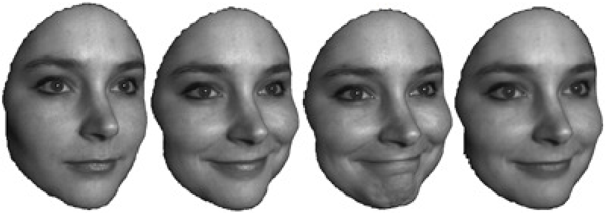
References

Main References

- RP+HW: *Noncommutative geometry, conformal geometry, and the local equivariant index theorem.* arXiv:1210.2032. Superceded by the 3 papers below.
- RP+HW: *Index map, σ-connections, and Connes-Chern character in the setting of twisted spectral triples.* arXiv:1310.6131. To appear in J. K-Theory.
- RP+HW: *Noncommutative geometry and conformal geometry. I. Local index formula and conformal invariants*. To be posted on arXiv soon.
- RP+HW: *Noncommutative geometry and conformal geometry. II. Connes-Chern character and the local equivariant index theorem*. To be posted on arXiv soon.
- RP+HW: *Noncommutative geometry and conformal geometry. III. Poincaré duality and Vafa-Witten inequality.* arXiv:1310.6138.

Conformal Geometry



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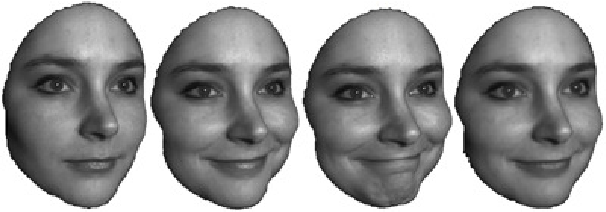

Overview of Noncommutative Geometry

Classical	**NCG**
Manifold M	Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$
Vector Bundle E over M	Projective Module $\mathcal{E}$ over $\mathcal{A}$ $\mathcal{E} = e\mathcal{A}^q, \quad e \in M_q(\mathcal{A}), \ e^2 = e$
de Rham Homology/Cohomology	Cyclic Cohomology/Homology
Atiyah-Singer Index Formula $\mathrm{ind} \not{D}_{\nabla^E} = \int \hat{A}(R^M) \wedge \mathrm{Ch}(F^E)$	Connes-Chern Character $\mathrm{Ch}(D)$ $\mathrm{ind}\, D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$
Characteristic Classes	Cyclic Cohomology for Hopf Algebras

Spectral Triples

Definition

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. $[D, a]$ is bounded for all $a \in \mathcal{A}$.

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Example (Dirac Spectral Triple)

$$(C^\infty(M), L^2_g(M, \$), \not{D}_g),$$

where (M^n, g) is a compact Riemanian spin manifold (n even), $\$ = \$^+ \oplus \$^-$ is the spinor bundle, and $\not{D}_{\hat{g}}$ is the Dirac operator.

Twisted Spectral Triples

Definition (Connes-Moscovici)

A twisted *spectral triple* $(\mathcal{A}, \mathcal{H}, D)_\sigma$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$ together with an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. $[D, a]_\sigma := Da - \sigma(a)D$ is bounded for all $a \in \mathcal{A}$.

Examples

Example (Conformal Deformation of Spectral Triples)

Given an ordinary spectral triple $(\mathcal{A}, \mathcal{H}, D)$, let $k \in \mathcal{A}$, $k > 0$. Then

$$(\mathcal{A}, \mathcal{H}, kDk)_\sigma, \qquad \sigma(a) = k^2 a k^{-2}, \ a \in \mathcal{A},$$

is a twisted spectral triple.

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Example (Conformal Change of Metric)

Let $(C^\infty(M), L^2_g(M, \not{S}), \not{D}_g)$ be a Dirac spectral triple. Consider the conformal change of metric,

$$\hat{g} = k^{-2} g, \qquad k \in C^\infty(M), \; k > 0.$$

Then $(C^\infty(M), L^2_{\hat{g}}(M, \not{S}), \not{D}_{\hat{g}})$ is unitarily equivalent to

$$(C^\infty(M), L^2_g(M, \not{S}), \sqrt{k}\not{D}_g\sqrt{k}).$$

Further Examples

Further Examples

- Conformal Dirac spectral triple (Connes-Moscovici).
- Twisted spectral triples over NC tori associated to conformal weights (Connes-Tretkoff).
- Poincaré duals of some ordinary spectral triples (RP+HW, Part 3).
- Twisted spectral triples associated to quantum statistical systems (e.g., Connes-Bost systems, supersymmetric Riemann gas) (Greenfield-Marcolli-Teh '13).

σ-Connections

Definition (Bimodule of σ-Differential Forms)

$$\Omega^1_{D,\sigma}(\mathcal{A}) = \mathrm{Span}\{a d_\sigma b;\ a, b \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H}),$$

where $d_\sigma b := [D, b]_\sigma = Db - \sigma(b)D$.

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Let $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$.

1. A *σ-translate* is a finitely generated projective module $\mathcal{E}^\sigma$ together with a linear isomorphism $\sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma$ such that

$$\sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \qquad \forall \xi \in \mathcal{E}\ \forall a \in \mathcal{A}.$$

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2. A *σ-connection* is a given by a σ-translate $\mathcal{E}^\sigma$ and a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma \otimes \Omega^1_{D,\sigma}(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi) \otimes d_\sigma a + \left(\nabla^{\mathcal{E}}\xi\right) a \qquad \forall a \in \mathcal{A}\ \forall \xi \in \mathcal{E}.$$

Operator $D_{\nabla^{\mathcal{E}}}$

Proposition (RP+HW)

1. *The data of a σ-connection $\nabla^{\mathcal{E}}$ defines a closed unbounded operator,*

$$D_{\nabla^{\mathcal{E}}} = \begin{pmatrix} 0 & D^{-}_{\nabla^{\mathcal{E}}} \\ D^{+}_{\nabla^{\mathcal{E}}} & 0 \end{pmatrix}, \quad D^{\pm}_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes \mathcal{H}^{\pm} \to \mathcal{E}^{\sigma} \otimes \mathcal{H}^{\mp}.$$

2. *The operators $D^{\pm}_{\nabla^{\mathcal{E}}}$ are Fredholm.*

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Remark

In all the main examples $\operatorname{ind} D_{\nabla^{\mathcal{E}}}$ is actually an integer.

Index Map and Connes-Chern Character

Proposition (Connes-Moscovici, RP+HW)

There is a unique additive map $\mathrm{ind}_{D,\sigma} : K_0(\mathcal{A}) \to \frac{1}{2}\mathbb{Z}$ *such that*

$$\mathrm{ind}_D[\mathcal{E}] = \mathrm{ind}\, D_{\nabla^{\mathcal{E}}} \qquad \forall (\mathcal{E}, \nabla^{\mathcal{E}}).$$

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Theorem (Connes-Moscovici, RP+HW)

Assume that $\mathrm{Tr}\,|D|^{-p} < \infty$ *for some* $p \geq 1$. *Then there is a (periodic) cyclic cohomology class* $\mathrm{Ch}(D)_\sigma \in \mathrm{HP}^0(\mathcal{A})$ *, called Connes-Chern character, such that*

$$\mathrm{ind}\, D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D)_\sigma, \mathrm{Ch}(\mathcal{E})\rangle \qquad \forall (\mathcal{E}, \nabla^{\mathcal{E}}),$$

where $\mathrm{Ch}(\mathcal{E})$ *is the Chern character in periodic cyclic homology.*

Conformal Dirac Spectral Triple

Setup

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3. G is a group of conformal diffeomorphisms preserving $\mathcal{C}$. Thus, given any metric $g \in \mathcal{C}$ and $\phi \in G$,

$$\phi_* g = k_\phi^{-2} g \text{ with } k_\phi \in C^\infty(M), \; k_\phi > 0.$$

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$$\phi_* g = k_\phi^{-2} g \text{ with } k_\phi \in C^\infty(M),\ k_\phi > 0.$$
4. $C^\infty(M) \rtimes G$ is the (discrete) crossed-product algebra, i.e.,
$$C^\infty(M) \rtimes G = \left\{ \sum f_\phi u_\phi;\ f_\phi \in C_c^\infty(M) \right\},$$
$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

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Lemma (Connes-Moscovici)

For $\phi \in G$ *define* $U_\phi : L^2_g(M, \$) \to L^2_g(M, \$)$ *by*

$$U_\phi \xi = k_\phi^{-\frac{n}{2}} \phi_* \xi \quad \forall \xi \in L^2_g(M, \$).$$

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Then U_ϕ is a unitary operator, and

$$U_\phi \not{D}_g U_\phi^* = \sqrt{k_\phi} \not{D}_g \sqrt{k_\phi}.$$

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Then U_ϕ *is a unitary operator, and*

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Proposition (Connes-Moscovici)

The datum of any metric $g \in \mathcal{C}$ *defines a twisted spectral triple* $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g \right)_{\sigma_g}$ *given by*

1. *The Dirac operator* $\not{D}_g$ *associated to* g.
2. *The representation* $fu_\phi \to fU_\phi$ *of* $C^\infty(M) \rtimes G$ *in* $L^2_g(M, \not{S})$.
3. *The automorphism* $\sigma_g(fu_\phi) := k_\phi^{-1} fu_\phi$.

Theorem (RP+HW)

1. *The Connes-Chern character* $\mathrm{Ch}(\not{D}_g)_{\sigma_g} \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ *is an invariant of the conformal structure* $\mathcal{C}$.

Conformal Connes-Chern Character

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1. *The Connes-Chern character* $\mathrm{Ch}(\not{D}_g)_{\sigma_g} \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ *is an invariant of the conformal structure* $\mathcal{C}$.
2. *For any cyclic homology class* $\eta \in \mathrm{HP}_0(C^\infty(M) \rtimes G)$, *the pairing,*

$$\langle \mathrm{Ch}(\not{D}_g)_{\sigma_g}, \eta \rangle,$$

is a conformal invariant.

Computation of $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$

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Strategy

1. Thanks to the conformal invariance we can choose any metric $g \in \mathcal{C}$ to compute $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$.
2. If the conformal structure $\mathcal{C}$ is nonflat, then it contains a G invariant metric $\mathcal{C}$.
3. If $g \in \mathcal{C}$ is G-invariant, then $\sigma_g = 1$, and so the conformal Dirac spectral triple $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g\right)_{\sigma_g}$ is an ordinary spectral triple.

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4. In this case, the Connes-Chern character is computed as a consequence of a new heat kernel proof of the local equivariant index theorem of Atiyah-Segal, Donelly-Patodi, Gilkey.

Notation

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Let $\phi \in G$. Then

- M^ϕ is the fixed-point set of ϕ; this is a disconnected sums of submanifolds,
  $M^\phi = \bigsqcup M_a^\phi, \quad \dim M_a^\phi = a$ (a even).
- $\mathcal{N}^\phi = (TM^\phi)^\perp$ is the normal bundle (vector bundle over M^ϕ).
- Over M^ϕ, with respect to $TM_{|M^\phi} = TM^\phi \oplus \mathcal{N}^\phi$, there are decompositions,

$$\phi' = \begin{pmatrix} 1 & 0 \\ 0 & \phi'_{|\mathcal{N}^\phi} \end{pmatrix}, \qquad \nabla^{TM} = \nabla^{TM^\phi} \oplus \nabla^{\mathcal{N}^\phi}.$$

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For any G-invariant metric $g \in \mathcal{C}$, the Connes-Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$ is represented by the periodic cyclic cocycle $\varphi = (\varphi_{2m})$ given by

$$\varphi_{2m}(f^0 u_{\phi_0}, \cdots, f^{2m} u_{\phi_{2m}}) =$$

$$\frac{(-i)^{\frac{n}{2}}}{(2m)!} \sum_{0 \leq a \leq n} (2\pi)^{-\frac{a}{2}} \int_{M_a^\phi} \hat{A}(R^{TM_a^\phi}) \wedge \nu_\phi \left(R^{\mathcal{N}^\phi}\right) \wedge f^0 d\hat{f}^1 \wedge \cdots \wedge d\hat{f}^{2m},$$

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where $\phi := \phi_0 \circ \cdots \circ \phi_{2m}$,

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where $\phi := \phi_0 \circ \cdots \circ \phi_{2m}$, and $\hat{f}^j := f^j \circ \phi_0^{-1} \circ \cdots \circ \phi_{j-1}^{-1}$,

Local Index Formula in Conformal Geometry

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where $\phi := \phi_0 \circ \cdots \circ \phi_{2m}$, and $\hat{f}^j := f^j \circ \phi_0^{-1} \circ \cdots \circ \phi_{j-1}^{-1}$, and

$$\hat{A}\left(R^{TM^\phi}\right) := \det{}^{\frac{1}{2}} \left[\frac{R^{TM^\phi}/2}{\sinh\left(R^{TM^\phi}/2\right)} \right],$$
$$\nu_\phi\left(R^{\mathcal{N}^\phi}\right) := \det{}^{-\frac{1}{2}} \left[1 - \phi'_{|N^\phi} e^{-R^{\mathcal{N}^\phi}} \right].$$

Remark

The n-th degree component of φ is given by

$$\varphi_n(f^0 u_{\phi_0}, \cdots, f^n u_{\phi_n}) = \begin{cases} \int_M f^0 d\hat{f}^1 \wedge \cdots \wedge d\hat{f}^n & \text{if } \phi_0 \circ \cdots \circ \phi_n = 1, \\ 0 & \text{if } \phi_0 \circ \cdots \circ \phi_n \neq 1. \end{cases}$$

This represents Connes' transverse fundamental class of M/G.

Notation

Let $\phi \in G$. Then

- $\langle\phi\rangle$ is the conjugation class of ϕ.
- $G_\phi = \{\psi \in G;\ \psi \circ \phi = \phi \circ \psi\}$ is the stabilizer of ϕ.
- $H^\bullet(M_a^\phi)$ is the G_ϕ-invariant cohomology of M_a^ϕ

Cyclic Homology of $C^\infty(M) \rtimes G$

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Theorem (Brylinski-Nistor, Crainic)

Along the conjugation classes of G,

$$\mathrm{HP}_\bullet(C^\infty(M)\rtimes G) \simeq \bigoplus_{\langle\phi\rangle\in\langle G\rangle} \bigoplus_{0\leq a\leq n} H^\bullet(M_a^\phi)^{G^\phi}.$$

Construction of Geometric Cycles

Proposition (Brylinski-Getzler, Crainic, RP+HW)

1. *To any G_ϕ-invariant closed diff. form ω on M_a^ϕ is naturally associated an even cyclic cycle η_ω on $C^\infty(M) \rtimes G$.*

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1. *To any G_ϕ-invariant closed diff. form ω on M_a^ϕ is naturally associated an even cyclic cycle η_ω on $C^\infty(M) \rtimes G$.*
2. *If $\omega = f^0 df^1 \wedge \cdots \wedge df^m$, then*

$$\eta_\omega = \sum_{\sigma \in \mathfrak{S}_m} \epsilon(\sigma) \tilde{f}^0 \otimes \tilde{f}^{\sigma(1)} \otimes \cdots \otimes \tilde{f}^{\sigma(m-1)} \otimes f^{\sigma(m)} u_\phi,$$

where $\tilde{f}^j$ is a suitable smooth extension of f^j to M.

Conformal Invariants

Theorem (RP+HW)

Let ω be as in the previous slide. For any metric $g \in \mathcal{C}$ define

$$I_g(\omega) = \langle \mathrm{Ch}(\not{D}_g)_{\sigma_g}, \eta_\omega \rangle.$$

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1. *$I_g(\omega)$ is an invariant of the conformal structure $\mathcal{C}$ depending only on the class of ω in $H^\bullet(M_a^\phi)^{G_\phi}$.*

Conformal Invariants

Theorem (RP+HW)

Let ω be as in the previous slide. For any metric $g \in \mathcal{C}$ define

$$I_g(\omega) = \langle \mathrm{Ch}(\not{D}_g)_{\sigma_g}, \eta_\omega \rangle.$$

Then

1. $I_g(\omega)$ *is an invariant of the conformal structure* $\mathcal{C}$ *depending only on the class of* ω *in* $H^\bullet(M_a^\phi)^{G_\phi}$.
2. *For any* G*-invariant metric* $g \in \mathcal{C}$*, we have*

$$I_g(\omega) = \int_{M_a^\phi} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi \left(R^{\mathcal{N}^\phi} \right) \wedge \omega.$$

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Then

1. *$I_g(\omega)$ is an invariant of the conformal structure $\mathcal{C}$ depending only on the class of ω in $H^\bullet(M_a^\phi)^{G_\phi}$.*
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Remark

The above invariants are not of the same type as those considered by S. Alexakis in his solution of the Deser-Swimmer conjecture.