"Conjugacies" between dynamical systems, and their crossed products.

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For compact infinite metric spaces X and Y, and for two minimal homeomorphism $\alpha\colon X \to X$ and $\beta\colon Y \to Y$, starting from information on crossed products $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(Y) \rtimes_\beta \mathbb{Z}$, what can we say about the relation between two dynamical systems (X, α) and (Y, β)?

Dictionary:

For crossed product C^*-algebras:

Simpleness, isomorphisms, structured isomorphisms, tracial spaces, etc..

For dynamical systems:

Minimality, Rokhlin dimension, invariant probability measures, induced (co)homology maps, (flip) conjugacy, weak conjugacy, orbit equivalence, etc..

Spoiler: The main thing to connect dynamical system side and crossed product side is to find the "right descriptions".

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Definition

Let X and Y be two compact metric spaces. Let (X, α) and (Y, β) be two dynamical systems. They are conjugate if there exists $\sigma \in \mathrm{Homeo}(X, Y)$ such that $\sigma \circ \alpha = \beta \circ \sigma$. That is, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X \\ \sigma \downarrow & & \downarrow \sigma \\ Y & \xrightarrow{\beta} & Y \end{array}$$

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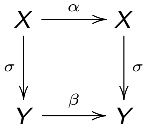

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Let X and Y be two compact metric spaces. Let (X, α) and (Y, β) be two dynamical systems. They are weakly (approximately) conjugate if there exist $\{\sigma_n \in Homeo(X, Y)\}$ and $\{\tau_n \in Homeo(Y, X)\}$, such that $\text{dist}(g \circ \beta, g \circ \tau_n^{-1} \circ \alpha \circ \tau_n) \to 0$ and $\text{dist}(f \circ \alpha, f \circ \sigma_n^{-1} \circ \beta \circ \sigma_n) \to 0$ for all $f \in C(X)$ and $g \in C(Y)$. Roughly speaking, the diagrams below "approximately" commute:

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Definition (Lin)

Let (X, α) and (Y, β) be two minimal dynamical systems. Assume that $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(Y) \rtimes_\beta \mathbb{Z}$ both have tracial rank zero. We say that (X, α) and (Y, β) are approximately K-conjugate if there exist homeomorphisms $\sigma_n : X \to Y$, $\tau_n : Y \to X$ and unital order isomorphisms $\rho : K_*(C(Y) \rtimes_\beta \mathbb{Z}) \to K_*(C(X) \rtimes_\alpha \mathbb{Z})$, such that

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Roughly speaking, approximate K-conjugacy $=$ weak (approximate) conjugacy $+$ "K-theoretic compatibility".

Definition

Let X be a compact metric space. For two minimal dynamical sytsems (X, α) and (X, β), we say that they are orbit equivalent if there exists a homeomorphism $F\colon X \to X$ such that $F(\text{orbit}_\alpha(x)) = \text{orbit}_\beta(F(x))$ for all $x \in X$. The map F is called an orbit map.

Definition (Giordano, Putnam, Skau)

Let (X, α) and (Y, β) be two minimal Cantor dynamical sytsems that are orbit equivalent. Two integer-valued functions $m, n\colon X \to \mathbb{Z}$ are called orbit cocyles associated with the orbit map F if $F \circ \alpha(x) = \beta^{n(x)} \circ F(x)$ and $F \circ \alpha^{m(x)}(x) = \beta \circ F(x)$ for all $x \in X$. We say that (X, α) and (Y, β) are strongly orbit equivalent if they are orbit equivalent and the orbit cocycles have at most one point of discontinuity.

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As for the crossed product C^*-algebras side, we first check whether they are classifiable. If so, we use the Elliott invariants to replace the original crossed products. Isomorphism of crossed products gives rise to isomorphism of the Elliott invariants, and we check how that is related to the dynamical system properties.

For example, for two irrational rotation algebras A_{θ_1} and A_{θ_2}, if they are isomorphic, we simply consider the following isomorphism:

$$(\mathbb{Z}+\theta_1\mathbb{Z},(\mathbb{Z}+\theta_2\mathbb{Z})_+,1)\longrightarrow(\mathbb{Z}+\theta_2\mathbb{Z},(\mathbb{Z}+\theta_2\mathbb{Z})_+,1).$$

Theorem (Giordano, Putnam, Skau)

For minimal Cantor dynamical systems (X, α) and (Y, β), $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(Y) \rtimes_\beta \mathbb{Z}$ are isomorphic if and only if (X, α) and (Y, β) are strongly orbit equivalent.

Remark: The proof uses the ordered Bratteli-Vershik model for the Cantor dynamics.

Fact: If the base space X is connected, then strong orbit equivalence is not a "good" definition. Besides, in case the base space is connected, orbit equivalence alone will simply imply flip conjugacy.

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Remark: The proof essentially follows the above mentioned "general strategy".

Remark: Rokhlin tower construction and the Berg technique are used to show the existence of the weak (approximate) conjugacies.

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The base space is $\mathbb{T}^1$

Isomorphism of crossed products implies that the two dynamical systems $(\mathbb{T}, \alpha)$ and $(\mathbb{T}, \beta)$ are weakly approximately conjugate (in fact, they are just flip conjugate). This comes from the Poincare classification theorem.

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(Result of Lin) Two Furstenberg transformations α and β on $\mathbb{T}^2$ are approximately K-conjugate if and only if the crossed product C^*-algebras are isomorphic.

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Isomorphism of crossed products might not imply the existence of weak (approximate) conjugacies.

Example (see Chris's 2002 arXiv paper): Minimal Furstenberg dynamical systems $(\mathbb{T}^3, \alpha)$ and $(\mathbb{T}^3, \beta)$, where

$$\alpha\colon (z_1, z_2, z_3) \mapsto (e^{2\pi i\theta} z_1, z_1^m z_2, z_2^n z_3) \text{ and } \beta\colon (z_1, z_2, z_3) \mapsto (e^{2\pi i\theta} z_1, z_1^n z_2, z_2^m z_3).$$

Classification result ensures that the two crossed product C^*-algebras are isomorphic. The induced maps (from α and β) on singular cohomology groups $H^1(\mathbb{T}^3; \mathbb{Z})$ $(\cong \mathbb{Z}^3)$ can be denoted as $\begin{pmatrix} 1 & m & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & n & 0 \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}$. Choose $m, n \in \mathbb{N} \setminus \{0\}$ such that these two matrices are not similar in $M_3(\mathbb{Z})$, which indicates that for all $\gamma \in Homeo(\mathbb{T}^3)$, α and $\gamma \circ \beta \circ \gamma^{-1}$ cannot be very close.

The base space is S^{2n+1}, $n \geq 1$

For uniquely ergodic homeomorphism on S^{2n+1} $(n \geq 1)$, by classification results of Winter, Lin and Niu, and due to Strung and Winter, we know that the crossed product C^*-algebras are classifiable. But the Elliott invariants do not contain much information.

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For bad cases with base space D, consider a new dynamical system with base $X \times D$, where X is the Cantor set. Due to the fact that X is totally disconnected, we might be able to recover weak (approximate) conjugacies on the new dynamical system.

For example, take base space to be $X \times \mathbb{T}^2$ (or $X \times \mathbb{T}^n$ in general), and consider the homeomorphisms such as

$$\alpha \times \varphi\colon (x, (t_1, t_2)) \mapsto (\alpha(x), \varphi_x((t_1, t_2))),$$

where $\alpha \in \text{Homeo}(X)$ and each φ_x is a Furstenberg transformation on $\mathbb{T}^2$.

Theorem (S)

Let $(X \times \mathbb{T}^2, \alpha \times \varphi)$ and $(X \times \mathbb{T}^2, \beta \times \psi)$ be two minimal dynamical systems such that all cocyle actions are Furstenberg transformations. Use A and B to denote these corresponding crossed product C^*-algebras. Suppose that $A \cong B$ and there exist $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ in Homeo(X) satisfying
1) $\deg(\varphi) = \deg(\psi) \circ \gamma_n$ for all $n \in \mathbb{N}$,
2) $\gamma_n \circ \alpha \circ \gamma_n^{-1} \to \beta$, $\sigma \circ \beta \circ \sigma_n^{-1} \to \alpha$.
Then $(X \times \mathbb{T}^2, \alpha \times \varphi)$ and $(X \times \mathbb{T}^2, \beta \times \psi)$ are weakly approximately conjugate.

For bad cases with base space D, consider a new dynamical system with base $X \times D$, where X is the Cantor set. Due to the fact that X is totally disconnected, we might be able to recover weak (approximate) conjugacies on the new dynamical system.

For example, take base space to be $X \times \mathbb{T}^2$ (or $X \times \mathbb{T}^n$ in general), and consider the homeomorphisms such as

$$\alpha \times \varphi \colon (x, (t_1, t_2)) \mapsto (\alpha(x), \varphi_x((t_1, t_2))),$$

where $\alpha \in \text{Homeo}(X)$ and each φ_x is a Furstenberg transformation on $\mathbb{T}^2$.

Theorem (S)

Let $(X \times \mathbb{T}^2, \alpha \times \varphi)$ and $(X \times \mathbb{T}^2, \beta \times \psi)$ be two minimal dynamical systems such that all cocyle actions are Furstenberg transformations. Use A and B to denote these corresponding crossed product C^*-algebras. Suppose that $A \cong B$ and there exist $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ in Homeo(X) satisfying
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As a given C^*-algebra might be realized as crossed product of minimal dynamical systems on different base spaces, C^*-algebra alone might be missing information on the base space. Instead of considering isomorphism on crossed product only, we assume the base space X is given and require one extra commutative diagram in K-theory:

$$\begin{array}{ccc} K_*(A) & \xrightarrow{\varphi} & K_*(B) \\ \uparrow{\scriptstyle \rho_A} & & \uparrow{\scriptstyle \rho_B} \\ K_*(C(X)) & \xrightarrow{\psi} & K_*(C(X)) \end{array}$$

This is the idea of augmented isomorphisms (by Lin & Matui).

Remark: For all the cases in the "Good news" part, isomorphism of crossed products automatically implies "augmented isomorphism".

Question: If we always start from "augmented isomorphisms" instead, can we get rid of the bad cases?

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