Decomposition of degenerate Gromov-Witten invariants

Joint with Q. Chen, M. Gross and B. Siebert

Dan Abramovich

Brown University

October 16, 2013

Hero:

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- 12

Motto:

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- thinking about 12 $\mapsto$ beautiful math

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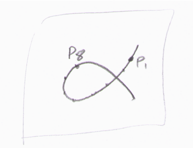
$12 =$ number of rational cubics through $p_1, \ldots, p_8$



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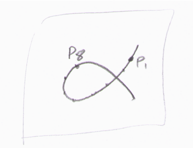

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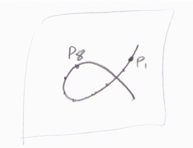

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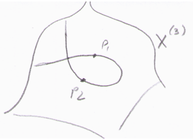
12 = number of rational curves in an elliptic pencil on a rational surface
specifically
12 = number of rational plane sections of $X^{(3)} \subset \mathbb{P}^3$ through p_1, p_2



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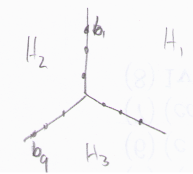
$$H_1 H_2 H_3 + tX^{(3)} = 0$$



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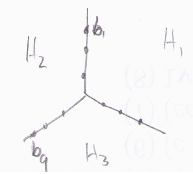

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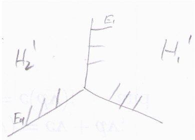

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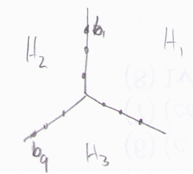

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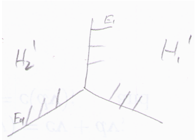

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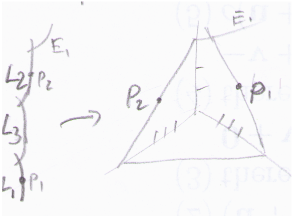
$$\begin{aligned}\int_{[\overline{\mathcal{M}}_b]^{\mathrm{virt}}} e^*\gamma &= \int_{[\overline{\mathcal{M}}]^{\mathrm{virt}}} e^*\gamma \cup \pi^* b = \int_{[\overline{\mathcal{M}}]^{\mathrm{virt}}} e^*\gamma \cup \pi^* 0 \\ &= \int_{[\overline{\mathcal{M}}_0]^{\mathrm{virt}}} e^*\gamma\end{aligned}$$

Back to example:

For each singular point b_i there is a plane H_i through p_1, p_2 and b_i



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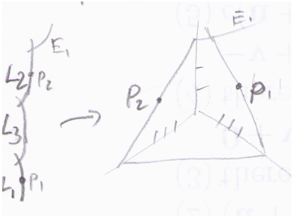

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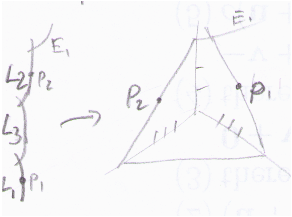

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$$12 = 9$$

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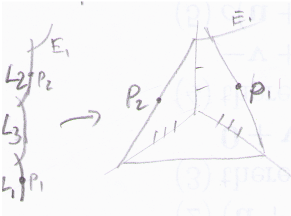
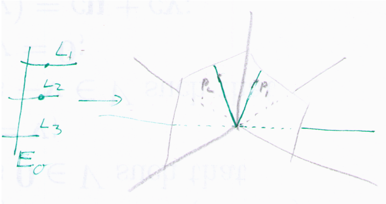


get

$$\begin{aligned} 12 &= 9 \quad \text{Anomaly?!?} \\ 12 &\stackrel{?}{=} 9 + \hbar(D \dots) \end{aligned}$$

The balance

There is a unique plane through p_1, p_2, O.



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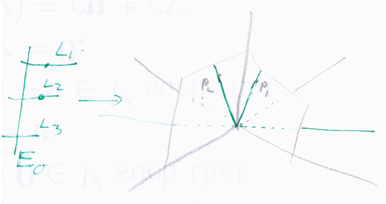

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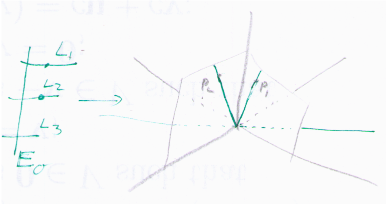

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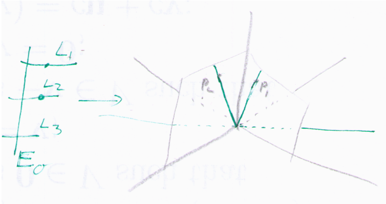

get

$$12 \quad = \quad 9 + 1 \times 3$$

What's with this multiplicity 3?

The decomposition formula

Theorem (ACGS)

$$[\overline{\mathcal{M}}_0]^{\text{virt}} \quad = \quad \sum_{f^t: G \to \Sigma_X} m_{f^t} [\overline{\mathcal{M}}_{f^t}]^{\text{virt}}$$

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- m_f *is the map* $N_{f^t} = \mathbb{N} \quad \to \quad N_B = \mathbb{N}$ *described below.*

Decomposing the example: the target

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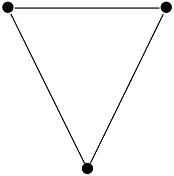
It is convenient to draw a slice $x + y + z = 1$:



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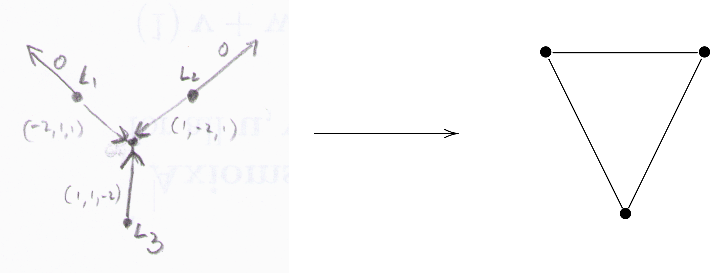

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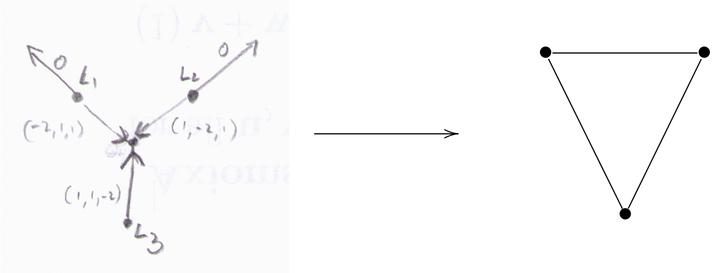

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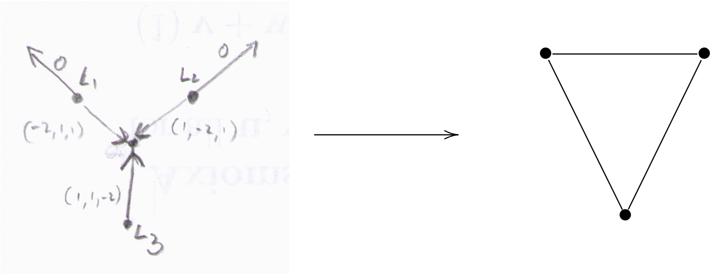

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- It lies over $x + y + z = 3 \in \Sigma_B$.

In general,

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & B \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} \Sigma_C & \xrightarrow{f^t} & \Sigma_X \\ \downarrow & & \downarrow \\ \Sigma_S & \longrightarrow & \Sigma_B \end{array}$$

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- Edges are labeled by slope vectors u_j.
- The fact that only finitely many u_j are possible given Γ is a nontrivial result.

The tropical moduli space

$$\overline{\mathcal{M}}^{\mathrm{trop}}_{f^t} =$$

$$\left\{ \big((v_i),(e_j)\big) \in \prod \sigma_i \times \prod \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{c} \forall\ v_{1,j} \xrightarrow{\quad q_j \quad} v_{2,j} \\ v_{2,j} - v_{1,j} = e_j u_{q_j} \end{array} \right\}.$$

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It evidently has the lattice

$$N_{f^t} = \left\{ ((v_i),(e_j)) \in \prod N_{\sigma_i} \times \prod \mathbb{N} \;\middle|\; \begin{array}{c} \forall\ v_{1,j} \xrightarrow{\quad q_j \quad} v_{2,j} \\ v_{2,j} - v_{1,j} = e_j u_{q_j} \end{array} \right\}.$$

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- m_{f^t} is determined by

$$\begin{array}{ccc} N_{f^t} & \longrightarrow & N_B \\ \| & & \| \\ \mathbb{N} & \xrightarrow[m_{f^t}]{} & \mathbb{N} \end{array}$$

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N.B. the example is unobstructed

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- Write $[\overline{\mathcal{M}}_0] = \sum_\tau m_\tau D_\tau$,

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Proof.

- Take a toric chart at the generic point of D_τ

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- $\mathcal{A}_X$ is locally like this.

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By the unobstructed case

$$[(\mathcal{A}_{\overline{\mathcal{M}}})_0] \quad = \quad \sum_{f^t} m_{f^t}[(\mathcal{A}_{\overline{\mathcal{M}}})_{f^t}].$$

We have another cartesian diagram

$$\begin{array}{ccc} \coprod m_{f^t}\overline{\mathcal{M}}_{f^t} & \xrightarrow{\Psi} & \overline{\mathcal{M}}_0 \\ \downarrow & & \downarrow \\ \coprod m_{f^t}(\mathcal{A}_{\overline{\mathcal{M}}})_{f^t} & \longrightarrow & (\mathcal{A}_{\overline{\mathcal{M}}})_0 \end{array}$$

We have another cartesian diagram

$$\begin{array}{ccc} \coprod m_{ft}\overline{\mathcal{M}}_{ft} & \xrightarrow{\Psi} & \overline{\mathcal{M}}_0 \\ \downarrow & & \downarrow \\ \coprod m_{ft}(\mathcal{A}_{\overline{\mathcal{M}}})_{ft} & \longrightarrow & (\mathcal{A}_{\overline{\mathcal{M}}})_0 \end{array}$$

A standard argument with obstructions shows that the theorem of Costello-Manolache applies,
so $\Psi_* \left[\coprod m_{ft}\overline{\mathcal{M}}_{ft}\right]^{\text{virt}} = [\overline{\mathcal{M}}_0]^{\text{virt}}$ as required.

Thank you for your attention.