STUDIES OF CLOSED/OPEN MIRROR SYMMETRY FOR QUINTIC THREE-FOLDS THROUGH LOG MIXED HODGE THEORY

0. Introduction

Fundamental Diagram

For classifying space D of MHS of specified type,

$$
\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
D_{\Sigma,\mathrm{val}} & \longleftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \longrightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
D_{\Sigma} & \longleftarrow & D^{\sharp}_{\Sigma} & & & &
\end{array}
$$

Hope to understand Hodge theoretic aspect of MS by this.

Mirror symmetry for quintic 3-folds

Mirror symmetry for A-model of quintic 3-fold V and B-model of its mirror V° was predicted in [CDGP91], and proved in following (1)–(3), which are equivalent.
Every statement is near large radius point q_0 of complexified Kähler moduli $\mathcal{KM}(V)$ and maximally unipotent monodromy point p_0 of complex moduli $\mathcal{M}(V^\circ)$.

$t := y_1/y_0$, $u := t/2\pi i$ and $q := e^t = e^{2\pi i u}$ from 3.3 below
and respective ones in 3.4 below.

(1) (*Potential.* [LLuY97]) $\Phi^{V}_{\mathrm{GW}}(t) = \Phi^{V^\circ}_{\mathrm{GM}}(t)$.

(2) (*Solutions.* [Gi96], [Gi97p])

$$\begin{aligned} J_{\mathcal{V}} &:= 5H\Big(1 + tH + \frac{d\Phi}{dt}\frac{H^2}{5} + \Big(t\frac{d\Phi}{dt} - 2\Phi\Big)\frac{H^3}{5}\Big) \\ I_{\mathcal{V}} &:= 5H(y_0 + y_1 H + y_2 H^2 + y_3 H^3) \end{aligned}$$

Then, $y_0 J_{\mathcal{V}} = I_{\mathcal{V}}$.

(3) (*Variation of Hodge structure.* [Morrison97])
$(q_0 \in \overline{\mathcal{KM}}(V)) \overset{\sim}{\leftarrow} (p_0 \in \overline{\mathcal{M}}(V^\circ))$ by canonical coordinate $q = \exp(2\pi i u)$, lifts over the punctured $\mathcal{KM}(V) \overset{\sim}{\leftarrow} \mathcal{M}(V^\circ)$ to

$$(\mathcal{H}^V, S, \nabla^{\mathrm{middle}}, \mathcal{H}^V_{\mathbf{Z}}, \mathcal{F}; 1, [\mathrm{pt}]) \overset{\sim}{\leftarrow} (\mathcal{H}^{V^\circ}, Q, \nabla^{\mathrm{GM}}, \mathcal{H}^{V^\circ}_{\mathbf{Z}}, \mathcal{F}; \tilde{\Omega}, g_0).$$

Our (4) below is equivalent to (1)–(3).

(4) (*Log period map*)
σ : monodromy cone transformed by a level structure into End of reference fiber of local system for A- and B- models.
Then, we have diagram of horizontal log period maps

$$\begin{array}{ccc} (q_0 \in \overline{\mathcal{KM}}(V)) & \overset{\sim}{\leftarrow} & (p_0 \in \overline{\mathcal{M}}(V^\circ)) \\ & \searrow \qquad \swarrow & \\ & ([\sigma, \exp(\sigma_{\mathbf{C}})F_0] \in \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma) & \end{array}$$

with extensions of specified sections in (3), where $(\sigma, \exp(\sigma_{\mathbf{C}})F_0)$ is nilpotent orbit and $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is fine moduli of LH of specified type.

(5) (*Inhomogenous solutions*, [Walcher07], [PSW08p], [MW09])
L: Picard-Fuchs differential operator for quintic mirror.

$$\mathcal{T}_A = \frac{u}{2} \pm \Big(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} n_d q^{d/2}\Big).$$

$$\mathcal{T}_B = \int_{C_-}^{C_+} \Omega, \quad \{C_\pm, \text{line}\} = \{x_1 + x_2 = x_3 + x_4 = 0\} \cap X_\psi.$$

$$L(y_0(z)\mathcal{T}_A(z)) = L(\mathcal{T}_B(z))(= \frac{15}{16\pi^2}\sqrt{z}) \quad (z = \frac{1}{\psi^5}).$$

In a neighborhood of MUM point p_0, we have the following (6).

(6) (*Computations of admissible normal function and domainwall tension on MUM point*)
$\mathcal{H}_{\mathbf{Q}} := \mathcal{H}_{\mathbf{Q}}^{V^\circ}, \quad \mathcal{T} := \mathcal{T}_B$
$L_{\mathbf{Q}}$: translation of local system $\mathbf{Q} \oplus \mathcal{H}_{\mathbf{Q}}$ by $\mathcal{T}e^0$ in $\mathcal{E}xt^1(\mathbf{Q}, \mathcal{H}_{\mathbf{Q}})$
$J_{L_{\mathbf{Q}}}$: Néron model for admissible normal function over $\mathcal{T}e^0$, whose weak fan is constructed in [KNU13p, Néron models for admissible normal functions]

$$\begin{array}{ccccccc} S := (z^{1/2}\text{-plane}) & \longrightarrow & J_{L_{\mathbf{Q}}} & \overset{\text{transl}}{\simeq} & \mathcal{H}_{\mathcal{O}}/(F^2 + \mathcal{H}_{\mathbf{Q}}) & \overset{\text{pol}}{\simeq} & (F^2)^*/\mathcal{H}_{\mathbf{Q}} \\ & & & & & & \downarrow \\ & & \bar{J}_{L_{\mathbf{Q}}} & \simeq & \mathcal{H}_{\mathcal{O}}/(F^1 + \mathcal{H}_{\mathbf{Q}}) & \simeq & (F^3)^*/\mathcal{H}_{\mathbf{Q}} \end{array}$$

To state following assertions, we use e^0, e^1 which are part of basis of $\mathcal{H}_{\mathcal{O}}$ respecting Deligne decomposition at p_0 (see 6 (2B)).

(6.1) $\mathcal{T}e^0$ as truncated normal function $S \to \bar{J}_{1,L_{\mathbf{Q}}}$.

(6.2) Truncated normal function in (6.1) uniquely lifts to admissible normal function $S \to J_{1,L_{\mathbf{Q}}}$.

(6.3) Followings are mirror:

$$0 \to H^4(V, \mathbf{Z}) \to H^4(V - Lg) \to H^2(Lg) \to 0$$

$$0 \to \mathbf{Z}e^1(\mathrm{gr}_2^M) \to \tfrac{1}{2}\mathbf{Z}e^1(\mathrm{gr}_2^M) \to (2\text{-torsion}) \to 0$$

Here Lg is real Lagrangian, and $M = M(N, W)$ around MUM point p_0.

(6.4) (5) tells that inverse of admissible normal function in (6.2) from its image is given by $16\pi^2/15$ times L applying to extension of $L_{\mathbf{Q}}$.

1. Log mixed Hodge theory

1.1. Category $\mathcal{B}(\log)$

S : subset of analytic space Z.
Strong topology of S in Z is strongest one among topologies on S s.t. for $\forall$ analytic space A and $\forall$ morphism $f : A \to Z$ with $f(A) \subset S$, $f : A \to S$ is continuous.

Log structure on local ringed space S is sheaf of monoids M on S and homomorphisim $\alpha : M \to \mathcal{O}_S$ s.t. $\alpha^{-1}\mathcal{O}_S^\times \xrightarrow{\sim} \mathcal{O}_S^\times$.

fs means finitely generated, integral and saturated.

Analytic space is call *log smooth* if, locally, it is isomorphic to open set of toric variety.

Log manifold is log local ringed space over $\mathbf{C}$ which has open covering $(U_\lambda)_\lambda$ satisfying:
For each λ, there exist log smooth fs log analytic space Z_λ, finite subset I_λ of global log differential 1-forms $\Gamma(Z_\lambda, \omega^1_{Z_\lambda})$, and isomorphism of log local ringed spaces over $\mathbf{C}$ between U_λ and open subset in strong topology of
$S_\lambda := \{z \in Z_\lambda \mid \text{image of } I_\lambda \text{ in stalk } \omega^1_z \text{ is zero}\}$ in Z_λ.

1.2. Ringed space $(S^{\log}, \mathcal{O}_S^{\log})$

$S \in \mathcal{B}(\log)$.

$S^{\log} := \{(s, h) \mid s \in S,\ h : M_s^{\mathrm{gp}} \to \mathbf{S}^1 \text{ hom. s.t. } h(u) = u/|u|\ (u \in \mathcal{O}_{S,s}^\times)\}$
endowed with weakest topology s.t. followings are continuous.

(1) $\tau : S^{\log} \to S, (s, h) \mapsto s$.

(2) For $\forall$open $U \subset S$ and $\forall f \in \Gamma(U, M^{\mathrm{gp}})$, $\tau^{-1}(U) \to \mathbf{S}^1$, $(s, h) \mapsto h(f_s)$.

τ is proper, surjective with $\tau^{-1}(s) = (\mathbf{S}^1)^{r(s)}$,
$r(s) := \mathrm{rank}(M^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ varies with $s \in S$.
Define $\mathcal{L}$ on $S^{\log}$ as fiber product:

$$\begin{array}{ccccc}
\mathcal{L} & \xrightarrow{\exp} & \tau^{-1}(M^{\mathrm{gp}}) & \ni & (f \text{ at } (s,h)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cont}(*, i\mathbf{R}) & \xrightarrow{\exp} & \mathrm{Cont}(*, \mathbf{S}^1) & \ni & h(f)
\end{array}$$

$\iota : \tau^{-1}(\mathcal{O}_S) \to \mathcal{L}$ is induced from

$$\begin{array}{ccccc} f & \in & \tau^{-1}(\mathcal{O}_S) & \xrightarrow{\exp} & \tau^{-1}(\mathcal{O}_S^\times) \subset \tau^{-1}(M^{\mathrm{gp}}) \\ \downarrow & & \downarrow & & \downarrow \\ (f-\bar{f})/2 & \in & \mathrm{Cont}(*, i\mathbf{R}) & \xrightarrow{\exp} & \mathrm{Cont}(*, \mathbf{S}^1) \end{array}$$

Define

$$\mathcal{O}_S^{\log} := \frac{\tau^{-1}(\mathcal{O}_S) \otimes \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L})}{(f \otimes 1 - 1 \otimes \iota(f) \mid f \in \tau^{-1}(\mathcal{O}_S))}.$$

Thus $\tau : (S^{\log}, \mathcal{O}_S^{\log}) \to (S, \mathcal{O}_S)$ as ringed spaces over $\mathbf{C}$.
For $s \in S$ and $t \in \tau^{-1}(s) \subset S^{\log}$, let $t_j \in \mathcal{L}_t$ $(1 \le j \le r(s))$ s.t. images in $(M^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ of $\exp(t_j)$ form a basis.
Then, $\mathcal{O}_{S,t}^{\log} = \mathcal{O}_{S,s}[t_j \ (1 \le j \le r(s)]$ is polynomial ring.

1.3. Toric variety

σ : nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$, i.e., sharp cone generated by finite number of mutually commutative nilpotent elements.
Γ : subgroup of $G_{\mathbf{Z}}$, and $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$.
Assume σ is generated over $\mathbf{R}_{\geq 0}$ by $\log \Gamma(\sigma)$.
$P(\sigma) := \Gamma(\sigma)^\vee = \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$.
$\mathrm{toric}_\sigma := \mathrm{Hom}(P(\sigma), \mathbf{C}^{\mathrm{mult}}) \supset \mathrm{torus}_\sigma := \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^\times)$,

$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^\times \to 1$ induces
$0 \to \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{Z}) \to \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}) \xrightarrow{\mathbf{e}} \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^\times) \to 1$,
where $\mathbf{e}(z \otimes \log \gamma) := e^{2\pi i z} \otimes \gamma$ ($z \in \mathbf{C}$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}} = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{Z})$).

$\rho \prec \sigma$ induces surjection $P(\rho) \leftarrow P(\sigma)$ hence open $\mathrm{toric}_\rho \hookrightarrow \mathrm{toric}_\sigma$.
$0_\rho \in \mathrm{toric}_\rho$ is $P(\rho) \to \mathbf{C}^{\mathrm{mult}}$; $1 \mapsto 1$, other elements of $P(\rho) \mapsto 0$.
$0_\rho \in \mathrm{toric}_\rho \subset \mathrm{toric}_\sigma$ by above open immersion.
Then, as set, $\mathrm{toric}_\sigma = \{\mathbf{e}(z)0_\rho \mid \rho \prec \sigma,\ z \in \sigma_{\mathbf{C}}/(\rho_{\mathbf{C}} + \log \Gamma(\sigma)^{\mathrm{gp}})\}$.

For $S := \mathrm{toric}_\sigma$, polar coordinate $\mathbf{R}_{\geq 0} \times \mathbf{S}^1 \to \mathbf{R}_{\geq 0} \cdot \mathbf{S}^1 = \mathbf{C}$ induces

$$\begin{aligned}
\tau : S^{\log} &= \mathrm{Hom}(P(\sigma), \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times \mathrm{Hom}(P(\sigma), \mathbf{S}^1) \\
&= \{(\mathbf{e}(iy)0_\rho, \mathbf{e}(x)) \mid \rho \prec \sigma,\ x \in \sigma_{\mathbf{R}}/(\rho_{\mathbf{R}} + \log \Gamma(\sigma)^{\mathrm{gp}}),\ y \in \sigma_{\mathbf{R}}/\rho_{\mathbf{R}}\} \\
&\to S = \mathrm{Hom}(P(\sigma), \mathbf{C}^{\mathrm{mult}}), \\
\tau(\mathbf{e}(iy)0_\rho, \mathbf{e}(x)) &= \mathbf{e}(x+iy)0_\rho.
\end{aligned}$$

By $0 \to \rho_{\mathbf{R}}/\log \Gamma(\rho)^{\mathrm{gp}} \to \sigma_{\mathbf{R}}/\log \Gamma(\sigma)^{\mathrm{gp}} \to \sigma_{\mathbf{R}}/(\rho_{\mathbf{R}} + \log \Gamma(\sigma)^{\mathrm{gp}}) \to 0$, $\tau^{-1}(\mathbf{e}(a+ib)0_\rho) = \{(\mathbf{e}(ib)0_\rho, \mathbf{e}(a+x)) \mid x \in \rho_{\mathbf{R}}/\log \Gamma(\rho)^{\mathrm{gp}}\} \simeq (\mathbf{S}^1)^r$, as set, where $r := \mathrm{rank}\,\rho$ varies with $\rho \prec \sigma$.

$H_\sigma = (H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$: canonical local system on $S^{\log}$ by representation $\pi_1(S^{\log}) = \Gamma(\sigma)^{\mathrm{gp}} \subset G_{\mathbf{Z}} = \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_w)_w)$.

1.4. Graded polarized LMH

$S \in \mathcal{B}(\log)$.
Pre-graded polarized log mixed Hodge structure on S is
$H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ consisting of
$H_{\mathbf{Z}}$: local system of $\mathbf{Z}$-free modules of finite rank on $S^{\log}$,
W : increasing filtration W of $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$,
$\langle\ ,\ \rangle_w$: nondegenerate $(-1)^w$-symmetric $\mathbf{Q}$-bilinear form on gr^W_w,
$H_{\mathcal{O}}$: locally free $\mathcal{O}_S$-module on S satisfying:
$\exists\ \mathcal{O}^{\log}_S \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}^{\log}_S \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ (*log Riemann-Hilbert correspondence*),
$\exists\ FH_{\mathcal{O}}$: decreasing filt. of $H_{\mathcal{O}}$ s.t. $F^pH_{\mathcal{O}}$, $H_{\mathcal{O}}/F^pH_{\mathcal{O}}$ locally free.
Put $F^p := \mathcal{O}^{\log}_S \otimes_{\mathcal{O}_S} F^pH_{\mathcal{O}}$. Then $\tau_*F^p = F^pH_{\mathcal{O}}$.
$\langle F^p(\mathrm{gr}^W_w), F^q(\mathrm{gr}^W_w)\rangle_w = 0\ (p + q > w)$.

Pre-GPLMH on S is *GPLMH on S* if its pullback to each $s \in S$ is GPLMH on s in the following sense.
Let $(H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ be a pre-GPLMH on log point s.

(1) (Admissibility) $\exists\ M(N, W)$ for $\forall$ logarithm N of local monodromy of local system $(H_{\mathbf{R}}, W, (\langle\ ,\ \rangle_w)_w)$.
(2) (Griffiths transversality) $\nabla F^p \subset \omega_s^{1,\log} \otimes F^{p-1}$, where $\omega_s^{1,\log}$ is log diff. 1-forms on $(s^{\log}, \mathcal{O}_s^{\log})$, $\nabla = d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_s^{\log} \otimes H_{\mathbf{Z}} \to \omega_s^{1,\log} \otimes H_{\mathbf{Z}}$.
(3) (Positivity) For $t \in s^{\log}$ and $\mathbf{C}$-alg. hom. $a : \mathcal{O}_{s,t}^{\log} \to \mathbf{C}$, $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ a filtration on $H_{\mathbf{C},t}$.
Then, $(H_{\mathbf{Z},t}(\mathrm{gr}_w^W), \langle\ ,\ \rangle_w, F(a))$ is PHS of weight w if a is sufficiently twisted: $|\exp(a(\log q_j))| \ll 1$ $(\forall j)$ for $(q_j)_{1 \le j \le n} \subset M_s$ which induce generators of $M_s/\mathcal{O}_s^\times$.

1.5. Nilpotent orbit

Fix $\Lambda := (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$, where
H_0 is free $\mathbf{Z}$-module of finite rank,
W is increasing filtration on $H_{0,\mathbf{Q}} := \mathbf{Q} \otimes H_0$,
$\langle\ ,\ \rangle_w$ is nondegenerate $(-1)^w$-symmetric form on gr^W_w,
$(h^{p,q})_{p,q}$ is set of Hodge numbers.

D : classifying space of GPMHS for data Λ, consisting of all Hodge filtrations.
$\check{D}$: "compact dual".
$G_A := \mathrm{Aut}(H_{0,A}, W, (\langle\ ,\ \rangle_w)_w)$,
$\mathfrak{g}_A := \mathrm{End}(H_{0,A}, W, (\langle\ ,\ \rangle_w)_w)$ $(A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C})$.

$\sigma \subset \mathfrak{g}_{\mathbf{R}}$: *nilpotent cone*, i.e., sharp cone generated by finite number of mutually commutative nilpotent elements.

$Z \subset \check{D}$ is *σ-nilpotent orbit* if (1)–(4) hold for $F \in Z$.

(1) $Z = \exp(\sigma_{\mathbf{C}})F$.

(2) $\exists\ M(N, W)$ for any $N \in \sigma$.

(3) $NF^p \subset F^{p-1}$ for any $N \in \sigma$ any p.

(4) If $N, \ldots, N_n$ generate σ and $y_j \gg 0$ $(\forall j)$, then $\exp(\sum_j iy_jN_j)F \in D$.

Weak fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ is set of nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$, defined over $\mathbf{Q}$, s.t.

(5) Every $\sigma \in \Sigma$ is admissible relative to W.

(6) If $\sigma \in \Sigma$ and $\tau \prec \sigma$, then $\tau \in \Sigma$.

(7) If $\sigma, \sigma' \in \Sigma$ have a common interior point and
if there exists $F \in \check{D}$ such that (σ, F) and (σ', F)
generate nilpotent orbits, then $\sigma = \sigma'$.

Let Σ be weak fan and Γ be subgroup of $G_{\mathbf{Z}}$.

Σ and Γ are *strongly compatible* if (8)–(9) hold:

(8) If $\sigma \in \Sigma$ and $\gamma \in \Gamma$, then $\mathrm{Ad}(\gamma)\sigma \in \Sigma$.

(9) For $\forall\ \sigma \in \Sigma$, σ is generated by $\log \Gamma(\sigma)$, where $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$.

1.6. Moduli of LMH of type Φ

$\Phi = (\Lambda, \Sigma, \Gamma)$: Λ is from 1.4, Σ weak fan and Γ subgroup of $G_{\mathbf{Z}}$ s.t. Σ and Γ are strongly compatible.

$\sigma \in \Sigma$. $S := \mathrm{toric}_\sigma$, $H_\sigma = (H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$ on $S^{\log}$.

Universal pre-GPLMH H on $\check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D}$ is given by H_σ together with isomorphism $\mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathbf{Z}} H_{\sigma,\mathbf{Z}} = \mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathcal{O}_{\check{E}_\sigma}} H_{\mathcal{O}}$, where $H_{\mathcal{O}} := \mathcal{O}_{\check{E}_\sigma} \otimes H_0$ is the free $\mathcal{O}_{\check{E}_\sigma}$-module coming from that on $\check{D}$ endowed with universal Hodge filtration F.

$E_\sigma := \{x \in \check{E}_\sigma \mid H(x) \text{ is a GPLMH}\}$.
Note that slits appear in E_σ because of log-point-wise Griffiths transversality 1.3 (2) and positivity 1.3 (3), or equivalently 1.4 (3) and 1.4 (4) respectively.

As set, $D_\Sigma := \{(\sigma, Z) \in \check{D}_{\mathrm{orb}} \mid \text{nilpotent orbit}, \sigma \in \Sigma, Z \subset \check{D}\}$.

Let $\sigma \in \Sigma$.
Assume Γ is neat.
Structure as object of $\mathcal{B}(\log)$ on $\Gamma\backslash D_\Sigma$ is introduced by diagram:

$$\begin{array}{ccc} E_\sigma & \overset{\text{GPLMH}}{\subset} & \check{E} := \text{toric}_\sigma \times \check{D} \\ \big\downarrow {\scriptstyle \sigma_{\mathbf{C}}\text{-torsor}} & & \\ \Gamma(\sigma)^{\text{gp}}\backslash D_\sigma & & \\ \big\downarrow {\scriptstyle \text{loc. isom.}} & & \\ \Gamma\backslash D_\Sigma & & \end{array}$$

Action of $h \in \sigma_{\mathbf{C}}$ on $(\mathbf{e}(a)0_\rho, F) \in E_\sigma$ is $(\mathbf{e}(h+a)0_\rho, \exp(-h)F)$, and projection is $(\mathbf{e}(a)0_\rho, F) \mapsto (\rho, \exp(\rho_{\mathbf{C}} + a)F)$.

$S \in \mathcal{B}(\log)$.

LMH of type Φ *on* S is a pre-GPLMH $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ endowed with Γ-level structure
$\mu \in H^0(S^{\log}, \Gamma \backslash \operatorname{Isom}((H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w), (H_0, W, (\langle\ ,\ \rangle_w)_w)))$
satisfying the following condition: For $\forall\ s \in S$, $\forall\ t \in \tau^{-1}(s) = s^{\log}$, $\forall$ representative $\tilde{\mu}_t : H_{\mathbf{Z},t} \xrightarrow{\sim} H_0$, $\exists\ \sigma \in \Sigma$ s.t. σ contains $\tilde{\mu}_t P_s \tilde{\mu}_t^{-1}$ and $(\sigma, \tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{S,t}} F_t))$ generates a nilpotent orbit.
Here $P_s := \operatorname{Image}(\operatorname{Hom}((M_S/\mathcal{O}_S^{\times})_s, \mathbf{N}) \hookrightarrow \pi_1(s^{\log}) \to \operatorname{Aut}(H_{\mathbf{Z},t}))$ is local monodromy monoid P_s of $H_{\mathbf{Z}}$ at s.
(Then, the smallest such σ exists.)

Theorem. (i) $\Gamma\backslash D_\Sigma \in \mathcal{B}(\log)$, *which is Hausdorff.*
If Γ *is neat,* $\Gamma\backslash D_\Sigma$ *is log manifold.*

(ii) *On* $\mathcal{B}(\log)$, $\Gamma\backslash D_\Sigma$ *represents functor* LMH_Φ *of LMH of type* Φ.

Log period map. *Let* $S \in \mathcal{B}(\log)$. *Then we have isomorphism*

$$\mathrm{LMH}_\Phi(S) \xrightarrow{\sim} \mathrm{Map}(S, \Gamma\backslash D_\Sigma), \; H \mapsto \big(S \ni s \mapsto [\sigma, \exp(\sigma_{\mathbf{C}})\tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}_{S,t}^{\log}} F_t)]$$

which is functorial in S.

Log period map is a unified compactification of period map and normal function of Griffiths.

3. Quintic threefolds

3.2. Quintic threefold and its mirror

V : general quintic 3-fold in $\mathbf{P}^4$.

$V_\psi : f := \sum_{j=1}^5 x_j^5 + \psi \prod_{j=1}^5 x_j = 0$ in $\mathbf{P}^4$ $\quad (\psi \in \mathbf{P}^1)$.
$G := \{(a_j) \in (\mu_5)^5 \mid a_1 \dots a_5 = 1\}$ acts V_ψ, $x_j \mapsto a_j x_j$.
V_ψ° : a crepant resolution of quotient singularity of V_ψ/G.
Devide further by action $(x_1, \dots, x_5) \mapsto (a^{-1}x_1, x_2, \dots, x_5)$ $(a \in \mu_5)$.

3.3. Picard-Fuchs equation on the mirror V°

Ω : holomorphic 3-form on V_ψ° induced from

$$\mathrm{Res}_{V_\psi}\left(\frac{\psi}{f}\sum_{j=1}^{5}(-1)^{j-1}x_j dx_1 \wedge \cdots \wedge (dx_j)^\wedge \wedge \cdots \wedge dx_5\right)$$

$z := 1/\psi^5$, $\delta := zd/dz$.

$L := \delta^4 + 5z(5\delta+1)(5\delta+2)(5\delta+3)(5\delta+4)$
is Picard-Fuchs differential operator for Ω, i.e., $L\Omega = 0$ via Gauss-Manin connection ∇.

$z = 0$: maximally unipotent monodromy point,
$z = \infty$: Gepner point,
$z = -5^{-5}$: conifold point.

$y_j\ (0 \leq j \leq 3)$: basis of solutions for L. $y_0 = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}(-z)^n$,
$y_1 = y_0 \log(-z) + 5\sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5}\Big(\sum_{j=n+1}^{5n} \frac{1}{j}\Big)(-z)^n$.

$t := y_1/y_0$, $u := t/2\pi i$: canonical parameters
$q := e^t = e^{2\pi i u}$: canonical coordinate, which is specific chart of log structure and gives mirror map.

$$\Phi_{\mathrm{GM}}^{V^\circ} = \frac{5}{2}\Big(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\Big) : \text{Gauss-Manin potential of } V_z^\circ.$$

$\tilde{\Omega} := \Omega/y_0$. Yukawa coupling at $z = 0$ is

$$Y := -\int_{V^\circ} \tilde{\Omega} \wedge \nabla_\delta \nabla_\delta \nabla_\delta \tilde{\Omega} = \frac{5}{(1+5^5 z)y_0(z)^2}\Big(\frac{q}{z}\frac{dz}{dq}\Big)^3.$$

3.4. A-model of the quintic V

$T_1 = H$: hyperplane section of V in $\mathbf{P}^4$
$K(V) = \mathbf{R}_{>0} T_1$: Kähler cone of V.
u : coordinate of $\mathbf{C}T_1$, $t := 2\pi i u$.
Complexified Kähler moduli is

$$\mathcal{KM}(V) := (H^2(V, \mathbf{R}) + iK(V))/H^2(V, \mathbf{Z}) \xrightarrow{\sim} \Delta^*,$$
$$uT_1 \mapsto q := e^{2\pi i u}.$$

$C \in H_2(V, \mathbf{Z})$: homology class of line on V.
$T^1 \in H^4(V, \mathbf{Z})$: Poincaré dual of C.
For $\beta = dC \in H_2(V, \mathbf{Z})$, define $q^\beta := q^{\int_\beta T^1} = q^d$.

Gromov-Witten potential of V is

$$\Phi_{\mathrm{GW}}^{V} := \frac{1}{6}\int_V (tT_1)^3 + \sum_{0\neq\beta\in H_2(V,\mathbf{Z})} N_d q^\beta = \frac{5t^3}{6} + \sum_{d>0} N_d q^d.$$

Here Gomov-Witten invariant N_d is

$$\overline{M}_{0,0}(\mathbf{P}^4,d) \xleftarrow{\pi_1} \overline{M}_{0,1}(\mathbf{P}^4,d) \xrightarrow{e_1} \mathbf{P}^4,$$
$$N_d := \int_{\overline{M}_{0,0}(\mathbf{P}^4,d)} c_{5d+1}(\pi_{1*}e_1^*\mathcal{O}_{\mathbf{P}^4}(5)).$$

$N_d = 0$ if $d \leq 0$.
$N_d = \sum_{k|d} n_{d/k}k^{-3}$, $n_{d/k}$ is instanton number.

3.5. Z-structure

B-model $\mathcal{H}^{V^\circ}$:
$f : X \to S^*$ family of quintic-mirrors over punctured nbd of p_0.
$\mathcal{H}^{V^\circ}_{\mathbf{Z}}$: extension of $R^3 f_* \mathbf{Z}$ over $S^{\log}$.
N : monodromy logarithm at p_0,
$W = W(N)$: monodromy weight filtration.
Define $W_{k,\mathbf{Z}} := W_k \cap \mathcal{H}^{V^\circ}_{\mathbf{Z}}$ for all k.

$b \in S^{\log}$: base point.
g_0, g_1, g_3, g_2 : symplectic $\mathbf{Z}$-basis of $\mathcal{H}^{V^\circ}_{\mathbf{Z}}(b)$ for cup product,
s.t. $g_0, \ldots, g_k$ generate $W_{2k}(b)$ for all k.

For $s \in \mathcal{O}^{\log}_S \otimes_{\mathcal{O}} \mathcal{H}^{V^\circ}_{\mathcal{O}}$, followings are equivalent.
(1) s belongs to $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$.
(2) $\nabla s = 0$ ($\nabla = \nabla^{\mathrm{GM}}$) and $s(b) \in \mathcal{H}^{V^\circ}_{\mathbf{Z}}(b)$ for some $b \in S^{\log}$.
(3) $\nabla s = 0$ and $s(\mathrm{gr}^W_k) \in \mathrm{gr}^W_{k,\mathbf{Z}}$ for $k := \min\{l \mid s \in \mathcal{O}^{\log}_S \otimes W_l\}$.

A-model $\mathcal{H}^V$:
$\nabla = \nabla^{\text{middle}}$: A-model connection from 3.6 (3A) below.
For $s \in \mathcal{O}_S^{\log} \otimes \mathcal{H}_{\mathcal{O}}^V$, define $s \in \mathcal{H}_{\mathbf{Z}}^V$ if $\nabla s = 0$ and $s(\mathrm{gr}_{2p}^W) \in H^{3-p,3-p}$ $(V, \mathbf{Z})$, $W_{2q} := \bigoplus_{l \le q} H^{3-l,3-l}(V)$, $p := \min\{q \mid s \in \mathcal{O}_S^{\log} \otimes W_{2q}\}$.

$0 \in S = \Delta$, $b \in \tau^{-1}(0) \subset S^{\log}$. $\mathcal{O}_{S,b}^{\log} = \mathcal{O}_{S,0}[t] = \mathbf{C}\{q\}[t]$: stalk at b.
$q = e^t = e^{2\pi i u}$, $u = x + iy$ with x, y real.

For $s \in \mathcal{O}_S^{\log} \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}^V$, followings are equivalent.
(4) s belongs to $\mathcal{H}_{\mathbf{Z}}^V$.
(5) $\nabla s = 0$ and $s(b) \in \mathcal{H}_{\mathbf{Z}}^V(b)$ for some $b \in S^{\log}$.
(6) $\nabla s = 0$ and, for fixed $x = 0$, limit as $y \to \infty$ of $\exp(iy(-N))s$ over $S^{\log}$ belongs to $\bigoplus_p H^{p,p}(V, \mathbf{Z})$.
(7) $\nabla s = 0$ and specialization $x \mapsto 0$ of limit of $\exp(iy(-N))s$ over $S^{\log}$ with x fixed and $y \to \infty$ belongs to $\bigoplus_p H^{p,p}(V, \mathbf{Z})$.

3.6. Correspondence table

We use $\Phi_{\mathrm{GW}}^{V} = \Phi_{\mathrm{GM}}^{V^\circ} =: \Phi$.

(1A) *Polarization of A-model of* V.

$$S(\alpha,\beta) := (-1)^p \int_V \alpha \cup \beta \quad (\alpha \in H^{p,p}(V), \beta \in H^{3-p,3-p}(V)).$$

(1B) *Polarization of B-model of* V°.

$$Q(\alpha,\beta) := (-1)^{3(3-1)/2} \int_{V^\circ} \alpha \cup \beta = -\int_{V^\circ} \alpha \cup \beta \quad (\alpha,\beta \in H^3(V^\circ)).$$

(2A) *Specified sections inducing* $\mathbf{Z}$*-basis of* gr^W *for A-model of* V.

$$T_0 := 1 \in H^0(V, \mathbf{Z}),\ T_1 := H \in H^2(V, \mathbf{Z}),$$
$$T^1 := C \in H^4(V, \mathbf{Z}),\ T^0 := [pt] \in H^6(V, \mathbf{Z}),$$

Then $S(T_0, T^0) = 1$ and $S(T_1, T^1) = -1$.
Hence T_0, T_1, $-T^0$, T^1 form symplectic base for S.

(2B) *Specified sections inducing* $\mathbf{Z}$*-basis of* gr^W *for B-model of* V°*.*

$$\mathcal{H}_{\mathcal{O}} = \bigoplus_p I^{p,p}, \quad \text{where} \quad I^{p,p} := \mathcal{W}_{2p} \cap \mathcal{F}^p \xrightarrow{\sim} \mathrm{gr}^{\mathcal{W}}_{2p}\,.$$

Since $N(\mathrm{gr}^W_{2p}) = 0$, gr^W_{2p} is a constant sheaf and hence

$$\mathrm{gr}^{\mathcal{W}}_{2p} \supset \mathrm{gr}^W_{2p} \supset (\mathrm{gr}^W_{2p})_{\mathbf{Z}} := W_{2p,\mathbf{Z}}/W_{2p-1,\mathbf{Z}}.$$

Take

$$e_0 := \tilde{\Omega} \in I^{3,3}, e_1 \in I^{2,2}, e^1 \in I^{1,1}, e^0 = g_0 \in I^{0,0}$$

inducing generators of $(\mathrm{gr}^W_{2p})_{\mathbf{Z}}$, and $Q(e_0, e^0) = 1$, $Q(e_1, e^1) = -1$. Hence e_0, e_1, $-e^0$, e^1 form symplectic base for Q.

(3A) *A-model connection* $\nabla = \nabla^{\mathrm{middle}}$ *of* V.

$$\nabla_\delta T^0 := 0, \quad \nabla_\delta T^1 := T^0,$$
$$\nabla_\delta T_1 := \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}T^1 = \Big(5 + \frac{1}{(2\pi i)^3}\frac{d^3\Phi_{\mathrm{hol}}}{du^3}\Big)T^1,$$
$$\nabla_\delta T_0 := T_1.$$

∇ is flat, i.e., $\nabla^2 = 0$.

(3B) *B-model connection* $\nabla = \nabla^{\mathrm{GM}}$ *of* V°.

$$\nabla_\delta e^0 = 0, \quad \nabla_\delta e^1 = e^0,$$
$$\nabla_\delta e_1 = \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}e^1 = Ye^1 = \frac{5}{(1+5^5)y_0(z)^2}\Big(\frac{q}{z}\frac{dz}{dq}\Big)^3 e^1,$$
$$\nabla_\delta e_0 = e_1.$$

(4A) ∇-*flat* $\mathbf{Z}$-*basis for* $\mathcal{H}^V_{\mathbf{Z}}$.

$$
\begin{aligned}
s^0 &:= T^0, \quad s^1 := T^1 - uT^0 = \exp(-uH)T^1, \\
s_1 &:= T_1 - \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}T^1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}T^0 \\
&= \exp(-uH)T_1 - \Big(\sum_{d>0}\frac{N_d d^2}{2\pi i}q^d\Big)T^1 + \Big(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\Big)T^0, \\
s_0 &:= T_0 - uT_1 + \frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\Big)T^1 - \frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big)T^0 \\
&= \exp(-uH)T_0 + \Big(\sum_{d>0}\frac{N_d d^2}{2\pi i}uq^d - \sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\Big)T^1 \\
&\quad - \Big(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}uq^d - \sum_{d>0}\frac{2N_d}{(2\pi i)^3}q^d\Big)T^0.
\end{aligned}
$$

(4B) ∇-*flat* $\mathbf{Z}$-*basis for* $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$.

$$s^0 := e^0, \quad s^1 := e^1 - ue^0,$$

$$s_1 := e_1 - \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}e^1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}e^0,$$

$$s_0 := e_0 - ue_1 + \frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\Big)e^1 - \frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big)e^0.$$

(5A) *Monodromy logarithm for A-model of V around q_0.*

$$Ns^0 = 0, \quad Ns^1 = -s^0, \quad Ns_1 = -5s^1, \quad Ns_0 = -s_1.$$

Matrix of monodromy logarithm N via basis s^0, s^1, s_1, s_0 coincides with matrix of cup product with $-H$ via basis T^0, T^1, T_1, T_0.

(5B) *Monodromy logarithm for B-model of V° around p_0.*

$$Ns^0 = 0, \quad Ns^1 = -s^0, \quad Ns_1 = -5s^1, \quad Ns_0 = -s_1.$$

$$(6A)\qquad T^0 = s^0, \quad T^1 = s^1 + us^0,$$

$$\begin{aligned} T_1 &= s_1 + \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}s^1 + \frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\Big)s^0, \\ &= \Big(s_1 + 5us^1 + \frac{5}{2}u^2s^0\Big) + \Big(\sum_{d>0}\frac{N_d d^2}{2\pi i}q^d\Big)s^1 \\ &\quad + \Big(\sum_{d>0}\frac{N_d d^2}{2\pi i}uq^d - \sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\Big)s^0 \end{aligned}$$

$$\begin{aligned} T_0 = 1_V &= s_0 + us_1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}s^1 + \frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big)s^0 \\ &= \Big(s_0 + us_1 + \frac{5}{2}u^2s^1 + \frac{5}{6}u^3s^0\Big) + \Big(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\Big)s^1 \\ &\quad + \Big(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}uq^d - 2\sum_{d>0}\frac{N_d}{(2\pi i)^3}q^d\Big)s^0. \end{aligned}$$

$$(6B) \qquad e^0 = s^0, \quad e^1 = s^1 + us^0,$$

$$e_1 = s_1 + \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}s^1 + \frac{1}{(2\pi i)^3}\Big(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\Big)s^0,$$

$$= \Big(s_1 + 5us^1 + \frac{5}{2}u^2s^0\Big) + \Big(\sum_{d>0}\frac{N_d d^2}{2\pi i}q^d\Big)s^1$$

$$+ \Big(\sum_{d>0}\frac{N_d d^2}{2\pi i}uq^d - \sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\Big)s^0$$

$$e_0 = \tilde{\Omega} = s_0 + us_1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}s^1 + \frac{1}{(2\pi i)^3}\Big(u\frac{d\Phi}{du} - 2\Phi\Big)s^0$$

$$= \Big(s_0 + us_1 + \frac{5}{2}u^2s^1 + \frac{5}{6}u^3s^0\Big) + \Big(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\Big)s^1$$

$$+ \Big(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}uq^d - 2\sum_{d>0}\frac{N_d}{(2\pi i)^3}q^d\Big)s^0$$

$$= s_0 + \frac{1}{2\pi i}\frac{y_1}{y_0}s_1 + \frac{5}{(2\pi i)^2}\frac{y_2}{y_0}s^1 + \frac{5}{(2\pi i)^3}\frac{y_3}{y_0}s^0.$$

3.9. Proof of (3) $\Rightarrow$ (4) in Introduction

Proof 1, by nilpotent orbit theorem.
$S^* := \mathcal{KM}(V) \subset S := \overline{\mathcal{KM}}(V)$ for A-model,
$S^* := \mathcal{M}(V^\circ) \subset S := \overline{\mathcal{M}}(V^\circ)$ for B-model.
S endowed with log structure associated to $S \smallsetminus S^*$.
VPHS on S^* with unipotent monodromy along $S \smallsetminus S^*$ extends uniquely to a LVPH on S by LH theoretic interpretation of nilpotent orbit theorem of Schmid.
$1 = T_0$ (resp. $[\mathrm{pt}] = T^0$) for A-model and
$\tilde{\Omega} = e_0$ (resp. $g_0 = e^0$) for B-model extend over S
as canonical extension (resp. invariant section). $\square$

Proof 2, by correspondence table in 3.6.

$$\begin{array}{ccc}
\tilde{S}^{\log} := \mathbf{R} \times i(0,\infty] & \supset & \tilde{S}^* := \mathbf{R} \times i(0,\infty) \\
\downarrow & & \downarrow \\
S^{\log} & \supset & S^* \\
\tau \downarrow & & \\
S & &
\end{array}$$

The coordinate u of $\tilde{S}^*$ extends over $\tilde{S}^{\log}$.
$u_0 := 0 + i\infty \in \tilde{S}^{\log} \mapsto b := \bar{0} + i\infty \in S^{\log} \mapsto q = 0 \in S$
which corresponds to q_0 for A-model and p_0 for B-model.

(a) $H_{\mathbf{Z}} := \mathcal{H}_{\mathbf{Z}}^{V}$ for A-model and $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ for B-model over S^* with respective symplectic basis $s_0, s_1, -s^0, s^1$ extends over $S^{\log}$ with extended symplectic basis.

Note that to fix a base point $u = u_0$ on $\tilde{S}^{\log}$ is equivalent to fix a base point b on $S^{\log}$ and also a branch of $(2\pi i)^{-1} \log q$.

(b) Regarding $H_0 := H_{\mathbf{Z},u_0} = H_{\mathbf{Z},b}$ as a constant sheaf on $S^{\log}$, we have an isomorphism $\mathcal{O}_S^{\log} \otimes H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes H_0$ of $\mathcal{O}_S^{\log}$-modules whose restriction induces $1 \otimes H_{\mathbf{Z},b} = 1 \otimes H_0$.

(c) $\tau_*(\mathcal{O}_S^{\log} \otimes H_{\mathbf{Z}})$ yields Deligne canonical extension of $H_{\mathcal{O}_{S^*}}$ over S. T_0, T_1, T^1, T^0 and e_0, e_1, e^1, e^0 yield monodromy invariant bases of $\mathcal{O}_{S^*}$-modules respecting Hodge filtration for each case.
These bases and hence Hodge filtrations extend over $q = 0$.

(c) follows from (1), (2), (3) below. $R := \mathcal{O}^{\log}_{S,b} = \mathbf{C}\{q\}[u]$.

(1) T_j, T^j and e_j, e^j are R-linear combinations of respective s_j, s^j.
(2) s_j, s^j are R-linear combinations of $s_j(b), s^j(b) \in H_{\mathbf{Z},b} = H_0$.
(3) Coefficients $h \in R$ of the composition of (1) and (2) are monodromy invariant holomorphic on S^* with $\lim_{q\to 0} qh = 0$. Hence, $q = 0$ is a removable singularity of h and value of h at $q = 0$ is determined.

Thus, PVHS $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, H_{\mathcal{O}})$ of type $(\Lambda, \Gamma(\sigma)^{\mathrm{gp}})$ over S^* extends to pre-PLH of type $\Phi = (\Lambda, \sigma, \Gamma(\sigma)^{\mathrm{gp}})$ over S, where $\sigma := \exp(\mathbf{R}_{\geq 0} N)$ with N from 0 (4). (Note that N here is $-N$ of N in Section 1.)

Admissibility is obvious in pure case.
Griffiths transversality follows from definitions of T_0, T_1, T^1, T^0, e_0, e_1, e^1, e^0, and ∇^{middle}, ∇^{GM}.
Positivity: We check for B-model. A-model is analogous.
$F_y := \exp(iy(-N))F(u_0) \in \check{D}$.
$v_3(y) := \exp(iy(-N))e_0(u_0)$ and $\exp(iy(-N))e_1(u_0)$ form basis of F_y^2 respecting F_y^3.
Compute basis $v_2(y)$ of $F_y^2 \cap \overline{F_y^1} = F_y^2 \cap (\overline{F_y^3})^\perp$ for Q.
Check that coefficients of highest terms in y of Hodge norms $i^3 Q(v_3(y), \overline{v_3(y)})$ and $iQ(v_2(y), \overline{v_2(y)})$ are both positive.
The extension of the specific sections has already seen. $\square$

4. Proof of (6) in Introduction

First announcement on Log Hodge Theory [KU99] was published in proceeding of CRM Summer School 1998, Banff.

We notice that we constructed complete fan Σ for classifying space D of polarized Hodge structure with $h^{p,q} = 1$ ($p + q = 3$, $p, q \geq 0$) as example in book [KU09], and also constructed weak fan which graphs any given admissible normal function over $\Gamma\backslash D_\Sigma$ in paper [KNU13p], appearing soon, in quite general setting.
In particular, Néron model $J_{L_{\mathbf{Q}}}$ in Intro (6) is already constructed.

In order to make monodromy of $\mathcal{T}$ around MUM point p_0 unipotent, we take double cover $z^{1/2}$.

Let $\mathcal{H} := \mathcal{H}^{V^\circ}$. We are looking for extension H

$$0 \to \mathcal{H} \to H \to \mathbf{Z} \to 0$$

of LMH with liftings $1_{\mathbf{Z}}$ and 1_F of $1 \in \mathbf{Z}$ respecting lattice and Hodge filtration, respectively.
Truncated normal function should be $\mathcal{T}$, i.e.,

$$Q(1_F - 1_{\mathbf{Z}}, \Omega) = \int_{C_-}^{C_+} \Omega = \mathcal{T},$$

where Q is polarization of $\mathcal{H}$.

To find such LMH, we use basis e_0, e_1, e^1, e^0 respecting Deligne decomp. of (M, F) from 3.6 (2B), ∇-flat integral basis s_0, s_1, s^1, s^0 from 3.6 (4B). We also use integral periods from 3.3 as $\eta_j := (2\pi i)^{-j} y_j$ for $j = 0, 1$ and $\eta_j := 5(2\pi i)^{-j} y_j$ for $j = 2, 3$.

First, translate trivial extension $(\mathrm{gr}^W)_{\mathbf{Q}} = \mathbf{Q} \oplus \mathcal{H}_{\mathbf{Q}}$ by $\mathcal{T}e^0$ and define $1_{\mathbf{Z}} := 1 + \mathcal{T}e^0$ to make local system $L_{\mathbf{Q}}$.

To find 1_F, write $1_F - 1_{\mathbf{Z}} = ae_0 + be_1 + ce_1 - \mathcal{T}e^0$ with $a, b, c \in \mathcal{O}^{\log}$. Griffiths transversality condition on $1_F - 1_{\mathbf{Z}}$ is understood as vanishing of coefficient of e^0 in $\nabla(1_F - 1_{\mathbf{Z}})$. Using 6 (3B), we have

$$\nabla_\delta(1_F - 1_{\mathbf{Z}}) = (\delta a)e_0 + (a + \delta b)e_1 + \Big(b\frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3} + \delta c\Big)e^1 + (c - \delta\mathcal{T})e^0.$$

Hence, above condition is equivalent to $c = \delta\mathcal{T}$ and a, b arbitrary. Using relation modulo F^2, we can take $a = b = 0$. Thus

$$1_F = 1_{\mathbf{Z}} + (\delta\mathcal{T})e^1 - \mathcal{T}e^0.$$

$(1_{\mathbf{Z}}, 1_F)$ is desired element in $\mathcal{E}xt^1_{\mathrm{LMH}}(\mathbf{Z}, \mathcal{H})$, and hence $1_F - 1_{\mathbf{Z}}$ is desired admissible normal function.

(6.1) and (6.2) are proved.

Next, we find splitting of weight filtration W of local system $L_{\mathbf{Q}}$.

Since mondromy of $\mathcal{T}$ around $p_0 : z^{1/2} = 0$, is $T_\infty^2(\mathcal{T}) = \mathcal{T} - \eta_0$ ([W07]), we flat it by $\mathcal{T} + \frac{1}{2}\eta_1$, which is written as $(\mathcal{T} + \frac{1}{2}\eta_1)s^0$ in $\mathcal{H}$, because $T_\infty^2(\eta_1) = \eta_0$.

But then, $\frac{1}{2}\eta_1$ is added to truncated normal function.
To solve this, using $e^1 = s^1 + us^0$ ($s^0 = e^0$, $u = \eta_1/\eta_0$), we modify it as

$$\frac{1}{2}\eta_0 s^1 + (\mathcal{T} + \frac{1}{2}\eta_1)s^0 = \frac{1}{2}\eta_0 e^1 + \mathcal{T} e^0.$$

This is desired splitting of W of local system $L_{\mathbf{Q}}$, and we define

$$1_{\mathbf{Z}}^{\mathrm{spl}} := 1 + \frac{1}{2}\eta_0 s^1 + (\mathcal{T} + \frac{1}{2}\eta_1)s^0 = 1 + \frac{1}{2}\eta_0 e^1 + \mathcal{T} e^0.$$

Lifting 1_F^{spl} for $1_{\mathbf{Z}}^{\mathrm{spl}}$ is computed as before, and we get

$$1_F^{\mathrm{spl}} = 1_{\mathbf{Z}}^{\mathrm{spl}} + (\delta\mathcal{T})e^1 - \mathcal{T}e^0.$$

$(1_{\mathbf{Z}}^{\mathrm{spl}}, 1_F^{\mathrm{spl}})$ is desired split element in $\mathcal{E}xt^1_{\mathrm{LMH}}(\mathbf{Z}, \mathcal{H})$.

Note that $1_F^{\mathrm{spl}} - 1_{\mathbf{Z}}^{\mathrm{spl}} = 1_F - 1_{\mathbf{Q}} = (\delta\mathcal{T})e^1 - \mathcal{T}e^0$.

For (6.3), recall that weight of A-model is reversed from degree of cohomology. Then it follows from

$$1_{\mathbf{Z}} - 1_{\mathbf{Z}}^{\mathrm{spl}} = -\frac{1}{2}(\eta_0 s^1 + \eta_1 s^0) == -\frac{1}{2}\eta_0 e^1.$$

(6.4) follow from definition of $1_{\mathbf{Z}}$ (or equivalently definition of $1_{\mathbf{Z}}^{\mathrm{spl}}$). In fact, from that we have $1_{\mathbf{Z}} - 1 = \frac{1}{2}\eta_0 s^1 + (\mathcal{T} + \frac{1}{2}\eta_1)s^0$ and hence $L(1_{\mathbf{Z}} - 1) = \frac{15}{16\pi^2} z^{1/2} s_0$.