Classical Mirror Constructions II

The Batyrev-Borisov Construction

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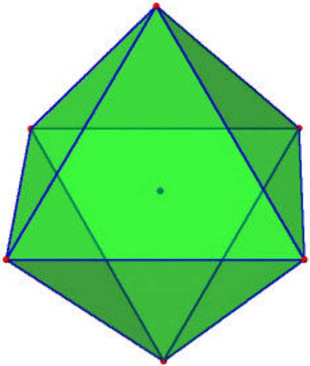
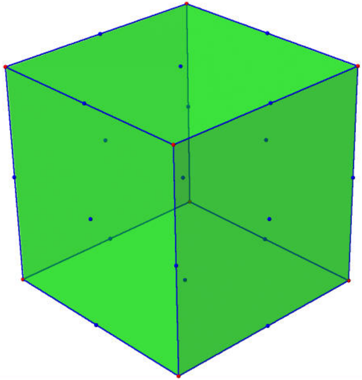
Outline

The Batyrev-Borisov Strategy

We can describe mirror families of Calabi-Yau manifolds using combinatorial objects called reflexive polytopes.

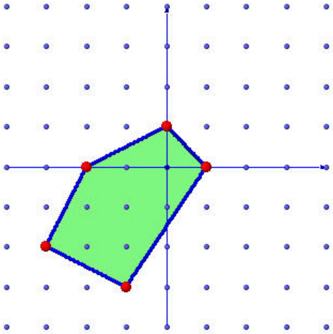




Lattice Polygons

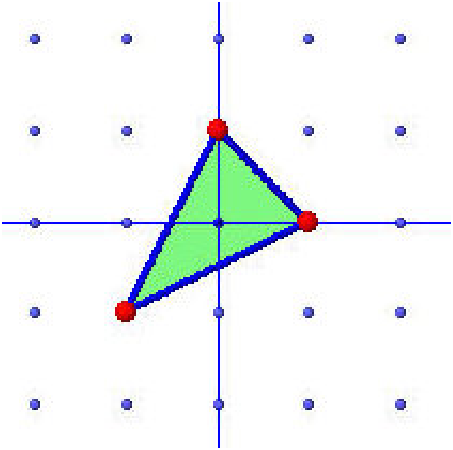
Let N be a lattice isomorphic to $\mathbb{Z}^2$.

A lattice polygon is a polygon in the plane $N_{\mathbb{R}}$ which has vertices in the lattice.

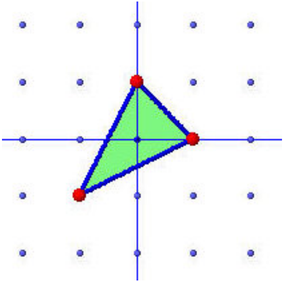


Fano Polygons

We say a lattice polygon is Fano if it has only one lattice point, the origin, in its interior.



Describing a Fano Polygon



- List the vertices

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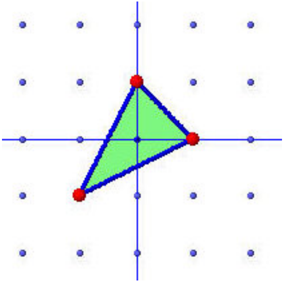
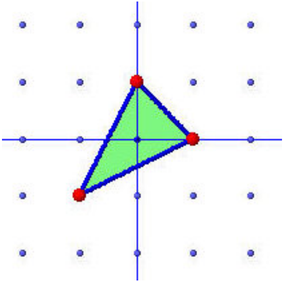


- List the vertices

$$\{(0,1),(1,0),(-1,-1)\}$$

Describing a Fano Polygon



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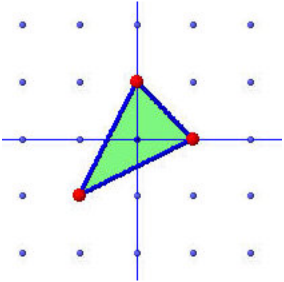
- List the equations of the edges

Describing a Fano Polygon



- List the vertices

$$\{(0,1),(1,0),(-1,-1)\}$$

- List the equations of the edges

$$\begin{aligned} -x - y &= -1 \\ 2x - y &= -1 \\ -x + 2y &= -1 \end{aligned}$$

A Dual Lattice

The dual lattice M of N is given by $\mathrm{Hom}(N, \mathbb{Z})$; it is also isomorphic to $\mathbb{Z}^2$. We write the pairing of $v \in N$ and $w \in M$ as $\langle v, w \rangle$. After choosing a basis, we may also use dot product notation:

$$(n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2$$

The pairing extends to a real-valued pairing on elements of $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$.

Polar Polygons

Edge equations define new polygons

Let Δ be a lattice polygon in $N_{\mathbb{R}}$ which contains $(0,0)$. The polar polygon Δ° is the polygon in $M_{\mathbb{R}}$ given by:

$$\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

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$$\begin{aligned} (x, y) \cdot (-1, -1) &= -1 \\ (x, y) \cdot (2, -1) &= -1 \\ (x, y) \cdot (-1, 2) &= -1 \end{aligned}$$

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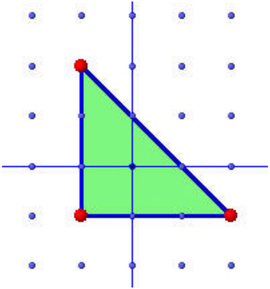


Mirror Pairs

If Δ is a Fano polygon, then:

- Δ° is a lattice polygon
- In fact, Δ° is another Fano polygon
- $(\Delta^\circ)^\circ = \Delta$.

We say that . . .

- Δ is a reflexive polygon.
- Δ and Δ° are a mirror pair.

A Polygon Duality

Mirror pair of triangles

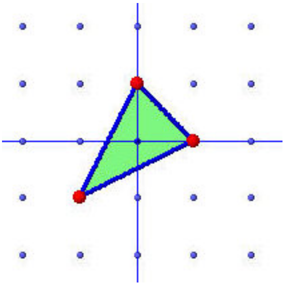


Figure: 3 boundary lattice points

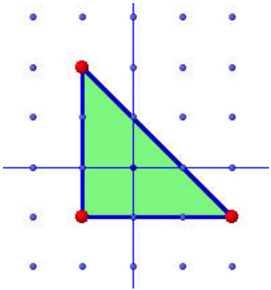


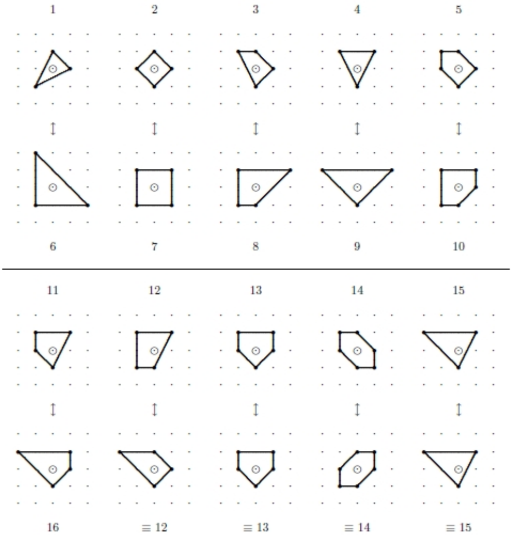
Figure: 9 boundary lattice points

$$3 + 9 = 12$$

Classifying Fano Polygons

- We can classify Fano polygons up to a change of coordinates that acts bijectively on lattice points
- There are 16 isomorphism classes of Fano polygons

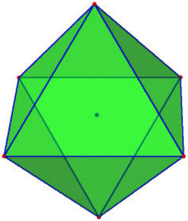
Mirror Pairs of Polygons



Other Dimensions

Definition

Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ be a set of points in $\mathbb{R}^k$. The polytope with vertices $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ is the convex hull of these points.



Polar Polytopes

Let $N \cong \mathbf{Z}^n$ be a lattice. A lattice polytope is a polytope in $N_{\mathbb{R}}$ with vertices in N.
As before, we have a dual lattice M and a pairing $\langle v, w \rangle$.

Definition

Let Δ be a lattice polytope in $N_{\mathbb{R}}$ which contains $(0, \ldots, 0)$. The polar polytope Δ° is the polytope in $M_{\mathbb{R}}$ given by:

$$\{(m_1, \ldots, m_k) : \langle (n_1, \ldots, n_k), (m_1, \ldots, m_k) \rangle \geq -1$$
$$\text{for all } (n_1, \ldots, n_k) \in \Delta\}$$

Reflexive Polytopes

Definition

A lattice polytope Δ is reflexive if Δ° is also a lattice polytope.

- If Δ is reflexive, $(\Delta^\circ)^\circ = \Delta$.
- Δ and Δ° are a mirror pair.

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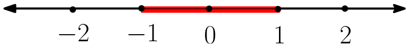

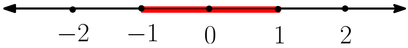

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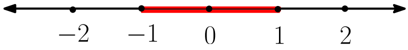
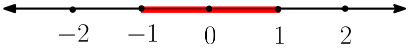
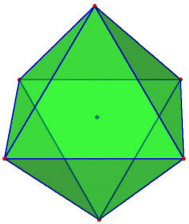
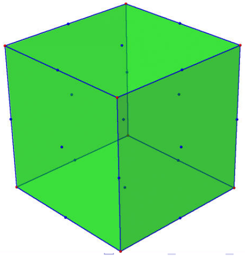


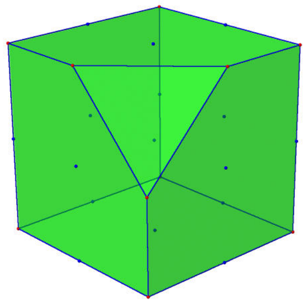






Fano vs. Reflexive

- Every reflexive polytope is Fano
- In dimensions $n \geq 3$, not every Fano polytope is reflexive



Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	
2	
3	
4	
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	
3	
4	
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
3	
4	
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
3	4,319
4	
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
3	4,319
4	473,800,776
5	

Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
3	4,319
4	473,800,776
5	??

Mirror Polytopes Yield Mirror Spaces

$$\begin{array}{ccc}
\text{polytope} & \longleftrightarrow & \text{polar polytope} \\
\downarrow & & \downarrow \\
\text{toric variety} & \longleftrightarrow & \text{polar toric variety} \\
\downarrow & & \downarrow \\
\text{hypersurface family} & \longleftrightarrow & \text{mirror hypersurface family}
\end{array}$$

Cones

A cone in N is a subset of the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$ generated by nonnegative $\mathbb{R}$-linear combinations of a set of vectors $\{v_1, \ldots, v_m\} \subset N$. We assume that cones are strongly convex, that is, they contain no line through the origin.

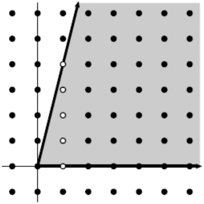


Figure: Cox, Little, and Schenk

Fans

A fan Σ consists of a finite collection of cones such that:

- Each face of a cone in the fan is also in the fan
- Any pair of cones in the fan intersects in a common face.

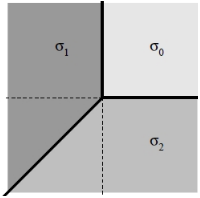


Figure: Cox, Little, and Schenk

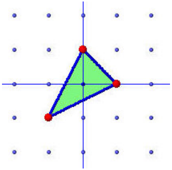
Simplicial fans

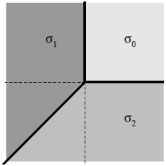
We say a fan Σ is simplicial if the generators of each cone in Σ are linearly independent over $\mathbb{R}$.

Fans from polytopes

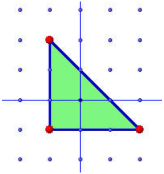
We may define a fan using a polytope in several ways:

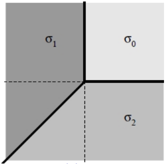
1. Take the fan R over the faces of $\diamond \subset N$.





2. Refine R by using other lattice points in $\diamond$ as generators of one-dimensional cones.
3. Take the normal fan S to $\diamond^\circ \subset M$.





Toric varieties as quotients

- Let Σ be a fan in $\mathbb{R}^n$.
- Let $\{v_1, \ldots, v_q\}$ be generators for the one-dimensional cones of Σ.
- Σ defines an n-dimensional toric variety V_Σ.
- V_Σ is the quotient of a subset $\mathbb{C}^q - Z(\Sigma)$ of $\mathbb{C}^q$ by a subgroup of $(\mathbb{C}^*)^q$.
- Each one-dimensional cone corresponds to a coordinate z_i on V_Σ.

Construction details: $Z(\Sigma)$

- Let $\mathcal{S}$ denote any subset of $\Sigma(1)$ that does *not* span a cone of Σ.
- Let $\mathcal{V}(\mathcal{S}) \subseteq \mathbb{C}^q$ be the linear subspace defined by setting $z_j = 0$ if the corresponding cone is in $\mathcal{S}$.
- $Z(\Sigma) = \cup_{\mathcal{S}} \mathcal{V}(\mathcal{S})$.

Construction details: $\ker(\phi)$

- $(\mathbb{C}^*)^q$ acts on $\mathbb{C}^q - Z(\Sigma)$ by coordinatewise multiplication.
- Write $v_j = (v_{j1}, \ldots, v_{jn})$
- Let $\phi : (\mathbb{C}^*)^q \to (\mathbb{C}^*)^n$ be given by

$$\phi(t_1, \ldots, t_q) \mapsto \left(\prod_{j=1}^{q} t_j^{v_{j1}}, \ldots, \prod_{j=1}^{q} t_j^{v_{jn}} \right)$$

The toric variety V_Σ associated with the fan Σ is given by

$$V_\Sigma = (\mathbb{C}^q - Z(\Sigma))/\mathrm{Ker}(\phi).$$

A Small Example

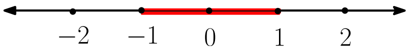


Figure: 1D Polytope $\diamond$

Let R be the fan obtained by taking cones over the faces of $\diamond$. $Z(\Sigma)$ consists of points of the form $(0,0)$.

$$V_R = (\mathbb{C}^2 - Z(\Sigma))/\sim$$

$$(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$$

where $\lambda \in \mathbb{C}^*$. Thus, $V_R = \mathbb{P}^1$.

Another Example

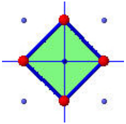


Figure: Polygon $\diamond$

Let R be the fan obtained by taking cones over the faces of $\diamond$. $Z(\Sigma)$ consists of points of the form $(0, 0, z_3, z_4)$ or $(z_1, z_2, 0, 0)$.

$$V_R = (\mathbb{C}^4 - Z(\Sigma))/ \sim$$

$$(z_1, z_2, z_3, z_4) \sim (\lambda_1 z_1, \lambda_1 z_2, z_3, z_4)$$
$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, \lambda_2 z_3, \lambda_2 z_4)$$

where $\lambda_1, \lambda_2 \in \mathbb{C}^*$. Thus, $V_R = \mathbb{P}^1 \times \mathbb{P}^1$.

Anticanonical Hypersurfaces

For each lattice point m in $\diamond^\circ$, choose a parameter α_m. Use this information to define a polynomial:

$$p_\alpha = \sum_{m \in M \cap \diamond^\circ} \alpha_m \prod_{j=1}^{q} z_j^{\langle v_j, m\rangle + 1}$$

Calabi-Yau Varieties

- If we use the fan R over the faces of $\diamond$ (or, equivalently, the normal fan to $\diamond^\circ$), p_α defines a Calabi-Yau variety.
- If we take a maximal simplicial refinement of R (using all the lattice points of $\diamond$), and $k \leq 4$, then p defines a smooth Calabi-Yau manifold V_α.
- Reversing the roles of $\diamond$ and $\diamond^\circ$ yields paired families of hypersurfaces.
- In particular, we can use pairs of 4-dimensional reflexive polytopes to define paired families of Calabi-Yau threefolds.

Toric Divisors

Each nonzero lattice point v_j in $\diamond$ defines a toric divisor, $z_j = 0$. We can intersect these divisors with V_α to yield elements of $H^{1,1}(V_\alpha)$.

- Not all of the toric divisors are independent.
- For general α, a divisor corresponding to the interior lattice point of a facet will not intersect V_α.
- The intersection of a toric divisor with V_α may "split" into several components.

Counting Kähler Moduli

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\diamond) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

- $\ell() =$ number of lattice points
- $\ell^*() =$ number of lattice points in the relative interior of a polytope or face
- The Γ are codimension 1 faces of $\diamond$
- The Θ are codimension 2 faces of $\diamond$
- $\hat{\Theta}$ is the face of $\diamond$ dual to Θ

Counting Complex Moduli

We know each lattice point in $\diamond^\circ$ corresponds to a monomial in p_α. For $k \geq 4$,

$$h^{d-1,1}(V_\alpha) = \ell(\diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

- $\ell()$ = number of lattice points
- $\ell^*()$ = number of lattice points in the relative interior of a polytope or face
- The Γ° are codimension 1 faces of $\diamond^\circ$
- The Θ° are codimension 2 faces of $\diamond^\circ$
- $\hat{\Theta}^\circ$ is the face of $\diamond$ dual to Θ°

Comparing V and V°

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\diamond) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

$$h^{d-1,1}(V_\alpha) = \ell(\diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

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$$h^{1,1}(V^\circ_\alpha) = \ell(\diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

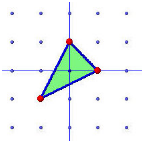
$$h^{d-1,1}(V^\circ_\alpha) = \ell(\diamond) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

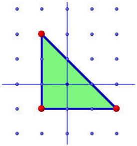
Mirror Symmetry from Mirror Polytopes

We have mirror families of Calabi-Yau varieties V_α and V_α° of dimension $d = k - 1$.

$$h^{1,1}(V_\alpha) = h^{d-1,1}(V_\alpha^\circ)$$
$$h^{d-1,1}(V_\alpha) = h^{1,1}(V_\alpha^\circ)$$

An Example

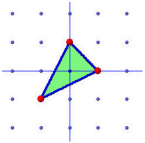


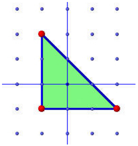


Four-dimensional analogue:

- $\diamond$ has vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(-1,-1,-1,-1)$.
- $\diamond^\circ$ has vertices $(-1,-1,-1,-1)$, $(4,-1,-1,-1)$, $(-1,4,-1,-1)$, $(-1,-1,4,-1)$, and $(-1,-1,-1,4)$.

An Example





Four-dimensional analogue:

- $\diamond$ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- $\diamond^\circ$ has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

$$h^{1,1}(V_\alpha) = \ell(\diamond) - n - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$
$$= 6 - 4 - 1 - 0 - 0 = 1.$$

Example (Continued)

- $\diamond$ has vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(-1,-1,-1,-1)$.
- $\diamond^\circ$ has vertices $(-1,-1,-1,-1)$, $(4,-1,-1,-1)$, $(-1,4,-1,-1)$, $(-1,-1,4,-1)$, and $(-1,-1,-1,4)$.

$$h^{1,1}(V_\alpha) = 1$$

$$\begin{aligned} h^{3-1,1}(V_\alpha) &= \ell(\diamond^\circ) - n - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ) \\ &= 126 - 4 - 1 - 20 - 0 = 101. \end{aligned}$$

The Hodge Diamond

Calabi-Yau Threefolds

V_α

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

V_α°

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Extrapolations

By looking more carefully at the structure of a reflexive polytope, one can study . . .

- Fibrations of Calabi-Yau varieties
- Degenerations of Calabi-Yau varieties
- Calabi-Yau complete intersections

Dolgachev's K3 Mirror Prescription

- Let X be a K3 surface.

$$H^2(X,\mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8 \oplus E_8$$

- If X_α is a family of K3 surfaces polarized by a lattice L, then the mirror family X_α° should be polarized by a lattice $\hat{L}$ such that

$$L^\perp = \hat{L} \oplus nU$$

- In particular, $\mathrm{rank}(L) + \mathrm{rank}(\hat{L}) = 20$.

Using Toric Divisors

Following Falk Rohsiepe, we observe ...

- We can intersect toric divisors with X_α to create a sublattice of $\mathrm{Pic}(X_\alpha)$
- We can compute the lattice pairings using purely combinatorial information about lattice points

Examining the Data

Set

$$\rho(\diamond) = \ell(\diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ).$$

$\diamond$	$\diamond^\circ$	$\rho(\diamond)$	$\rho(\diamond^\circ)$
0	4311	1	19
1	4281	4	18
2	4317	1	19
3	4283	2	18
4	4286	2	18
5	4296	2	18
8	3313	9	17

A Toric Correction Term

Set

$$\delta(\diamond) = \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ).$$

$\diamond$	$\diamond^\circ$	$\rho(\diamond)$	$\rho(\diamond^\circ)$	$\delta(\diamond)$
0	4311	1	19	0
1	4281	4	18	2
2	4317	1	19	0
3	4283	2	18	0
4	4286	2	18	0
5	4296	2	18	0
8	3313	9	17	6

Rohsiepe's Formulation

- Let $\diamond$ and $\diamond^\circ$ be a mirror pair of 3-dimensional reflexive polytopes, and let X_α and X_α° be the corresponding families of K3 surfaces.
- Write $i : X_\alpha \to W$ be the inclusion in the ambient toric variety, and let D_j be the toric divisors.
- Let L be the sublattice of $\mathrm{Pic}(X_\alpha)$ generated by $i^*(D_j)$
- Let $\hat{L}$ be the sublattice of $\mathrm{Pic}(X_\alpha^\circ)$ generated by all of the components of the intersections $D_j \cap X_\alpha^\circ$
- $$L^\perp = \hat{L} \oplus U$$

Some Picard rank 19 families

- Hosono, Lian, Oguiso, Yau:

$$x + 1/x + y + 1/y + z + 1/z - \Psi = 0$$

- Verrill:

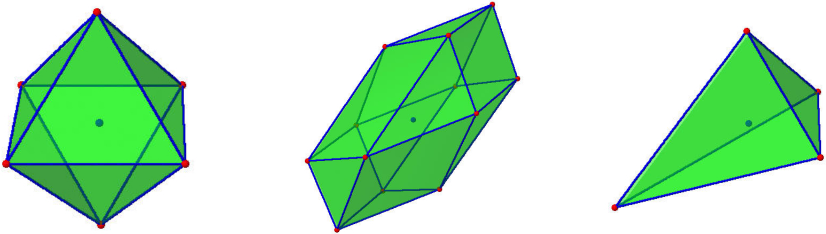
$$(1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)(xyz)$$

- Narumiya-Shiga:

$$Y_0 + Y_1 + Y_2 + Y_3 - 4tY_4$$
$$Y_0 Y_1 Y_2 Y_3 - Y_4^4$$

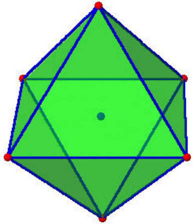
Toric realizations of the rank 19 families

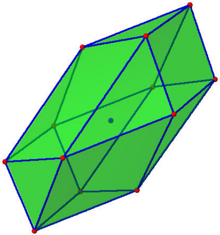
The polar polytopes $\diamond^\circ$ for [HLOY04], [V96], and [NS01].

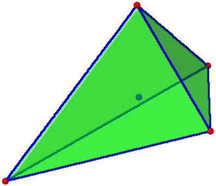


$$f(t) = \left(\sum_{x \,\in\, \mathrm{vertices}(\diamond^\circ)} \prod_{k=1}^{q} z_k^{\langle v_k, x\rangle + 1} \right) + t \prod_{k=1}^{q} z_k .$$

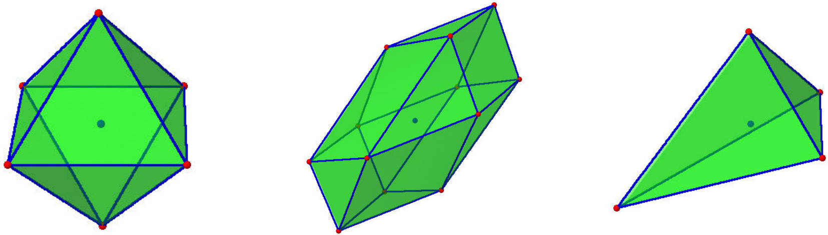
What do these polytopes have in common?





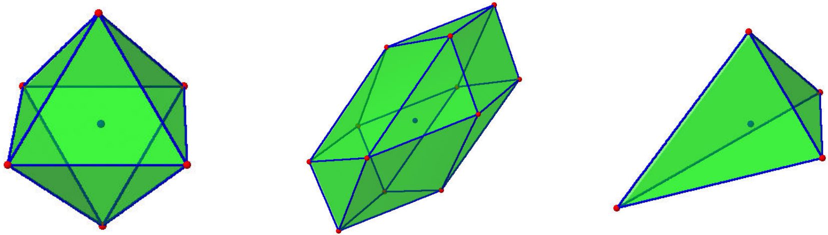


What do these polytopes have in common?



- The only lattice points of these polytopes are the vertices and the origin.

What do these polytopes have in common?



- The only lattice points of these polytopes are the vertices and the origin.
- The group G of orientation-preserving symmetries of the polytope acts transitively on the vertices.

Another symmetric polytope

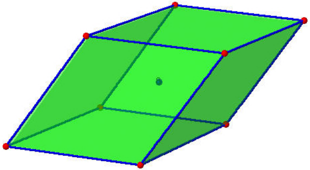


Figure: The skew cube

$$f(t) = \left(\sum_{x \in \text{vertices}(\diamond^\circ)} \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1} \right) + t \prod_{k=1}^{q} z_k .$$

Dual rotations

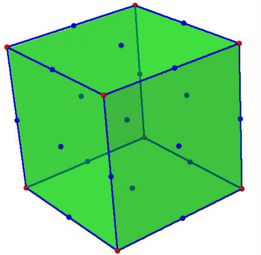


Figure: $\diamond$

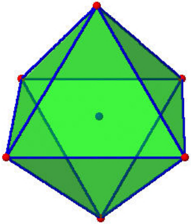


Figure: $\diamond^\circ$

We may view a rotation as acting either on $\diamond$ (inducing automorphisms on X_t) or on $\diamond^\circ$ (permuting the monomials of $f(t)$).

Symplectic Group Actions

Let G be a finite group of automorphisms of a K3 surface. For $g \in G$,

$$g^*(\omega) = \rho\omega$$

where ρ is a root of unity.

Definition

We say G acts *symplectically* if

$$g^*(\omega) = \omega$$

for all $g \in G$.

A subgroup of the Picard group

Definition

$$S_G = ((H^2(X, \mathbb{Z})^G)^\perp$$

Theorem ([N80a])

S_G is a primitive, negative definite sublattice of $\mathrm{Pic}(X)$.

The rank of S_G

Lemma

- *If X admits a symplectic action by the permutation group $G = \mathcal{S}_4$, then $\mathrm{Pic}(X)$ admits a primitive sublattice S_G which has rank* 17.
- *If X admits a symplectic action by the alternating group $G = \mathcal{A}_4$, then $\mathrm{Pic}(X)$ admits a primitive sublattice S_G which has rank* 16.

Why is the Picard rank 19?

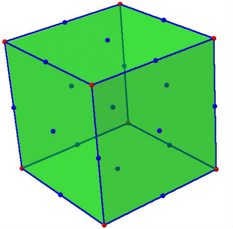


Figure: $\diamond$

We can use the orbits of G on $\diamond$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$.

Why is the Picard rank 19?

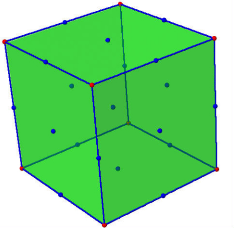


Figure: $\diamond$

We can use the orbits of G on $\diamond$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$.

- For the families of [HLOY04] and [V96], and the family defined by the skew cube, we conclude that $17 + 2 = 19$.
- For the family of [NS01], we conclude that $16 + 3 = 19$.

Collaborators

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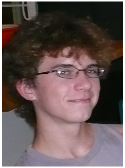
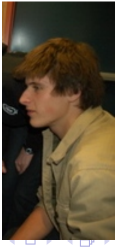
- Jacob Lewis (Universität Wien)
- Daniel Moore (HMC '11)
- Dmitri Skjorshammer (HMC '11)
- Ursula Whitcher (UWEC)




K3 surfaces from elliptic curves

Let E_1 and E_2 be elliptic curves, and let $A = E_1 \times E_2$.

- The Kummer surface $Km(A)$ is the minimal resolution of $A/\{\pm 1\}$.
- The Shioda-Inose surface $SI(A)$ is the minimal resolution of $Km(A)/\beta$, where β is an appropriately chosen involution.

Picard-Fuchs equations

- A period is the integral of a differential form with respect to a specified homology class.
- Periods of holomorphic forms encode the complex structure of varieties.
- The Picard-Fuchs differential equation of a family of varieties is a differential equation that describes the way the value of a period changes as we move through the family.
- Solutions to Picard-Fuchs equations for holomorphic forms on Calabi-Yau varieties define the mirror map.

Picard-Fuchs equations for rank 19 families

Let M be a free abelian group of rank 19, and suppose $M \hookrightarrow \mathrm{Pic}(X_t)$.

- The Picard-Fuchs equation is a rank 3 ordinary differential equation.
- The coefficients of the Picard-Fuchs equation are rational functions.
- The equation is Fuchsian (the singularities of the rational functions are controlled).

Symmetric Squares

- Let $L(y)$ be a homogeneous linear differential equation with coefficients in $\mathbb{C}(t)$.
- There exists a homogeneous linear differential equation $M(y) = 0$ with coefficients in $\mathbb{C}(t)$, such that . . .
- The solution space of $M(y)$ is the $\mathbb{C}$-span of

$$\{\nu_1\nu_2 \mid L(\nu_1) = 0 \text{ and } L(\nu_2) = 0\} .$$

Definition

$M(y)$ is the symmetric square of L.

Symmetric Square Formula

The symmetric square of the differential equation

$$a_2\frac{\partial^2 A}{\partial t^2} + a_1\frac{\partial A}{\partial t} + a_0 A = 0$$

is

$$a_2^2\frac{\partial^3 A}{\partial t^3} + 3a_1 a_2\frac{\partial^2 A}{\partial t^2} + (4a_0 a_2 + 2a_1^2 + a_2 a_1' - a_1 a_2')\frac{\partial A}{\partial t} + \\ (4a_0 a_1 + 2a_0' a_2 - 2a_0 a_2')A = 0$$

where primes denote derivatives with respect to t.

Picard-Fuchs equations and symmetric squares

Theorem

*[D00, Theorem 5] The Picard-Fuchs equation of a family of rank-*19 *lattice-polarized K3 surfaces can be written as the symmetric square of a second-order homogeneous linear Fuchsian differential equation.*

Quasismooth and regular hypersurfaces

Let Σ be a simplicial fan, and let X be a hypersurface in V_Σ. Suppose that X is described by a polynomial f in homogeneous coordinates.

Definition

If the derivatives $\partial f / \partial z_i$, $i = 1 \ldots q$ do not vanish simultaneously on X, we say X is quasismooth.

Quasismooth and regular hypersurfaces

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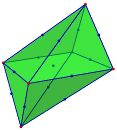
Definition

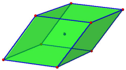
If the derivatives $\partial f/\partial z_i$, $i = 1 \dots q$ do not vanish simultaneously on X, we say X is quasismooth.

Definition

If the products $z_i \partial f/\partial z_i$, $i = 1 \dots q$ do not vanish simultaneously on X, we say X is regular and f is nondegenerate.

The Skew Octahedron





- Let $\diamond$ be the reflexive octahedron shown above.
- $\diamond$ contains 19 lattice points.
- Let R be the fan obtained by taking cones over the faces of $\diamond$. Then R defines a toric variety $V_R \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.
- Consider the family of K3 surfaces X_t defined by $f(t) = \left(\sum_{x \,\in\, \mathrm{vertices}(\diamond^\circ)} \prod_{k=1}^{q} z_k^{\langle v_k, x\rangle + 1}\right) + t \prod_{k=1}^{q} z_k$.
- X_t are generally quasismooth but not regular.

The Picard-Fuchs equation

Theorem ([KLMSW10])

Let $A = \int \mathrm{Res}\left(\frac{\Omega_0}{f}\right)$*. Then* A *is the period of a holomorphic form on* X_t*, and* A *satisfies the Picard-Fuchs equation*

$$\frac{\partial^3 A}{\partial t^3} + \frac{6(t^2-32)}{t(t^2-64)}\frac{\partial^2 A}{\partial t^2} + \frac{7t^2-64}{t^2(t^2-64)}\frac{\partial A}{\partial t} + \frac{1}{t(t^2-64)}A = 0.$$

As expected, the differential equation is third-order and Fuchsian.

Symmetric square root

The symmetric square root of our Picard-Fuchs equation is:

$$\frac{\partial^2 A}{\partial t^2} + \frac{(2t^2 - 64)}{t(t^2 - 64)}\frac{\partial A}{\partial t} + \frac{1}{4(t^2 - 64)}A = 0.$$

Semiample hypersurfaces

- Let R be a fan over the faces of a reflexive polytope
- Let Σ be a refinement of R
- We have a proper birational morphism $\pi : V_\Sigma \to V_R$
- Let Y be an ample divisor in V_R, and suppose $X = \pi^*(Y)$

Then X is semiample:

Definition

We say that a Cartier divisor D is *semiample* if D is generated by global sections and the intersection number $D^n > 0$.

The residue map

We will use a residue map to describe the cohomology of a K3 hypersurface X:

$$\mathrm{Res} : H^3(V_\Sigma - X) \to H^2(X).$$

Anvar Mavlyutov showed that Res is well-defined for quasismooth, semiample hypersurfaces in simplicial toric varieties.

Two ideals

Definition

The Jacobian ideal $J(f)$ is the ideal of $\mathbb{C}[z_1, \ldots, z_q]$ generated by the partial derivatives $\partial f / \partial z_i$, $i = 1 \ldots q$.

Definition

[BC94] The ideal $J_1(f)$ is the ideal quotient

$$\langle z_1 \partial f / \partial z_1, \ldots, z_q \partial f / \partial z_q \rangle : z_1 \cdots z_q.$$

The induced residue map

Let Ω_0 be a holomorphic 3-form on V_Σ. We may represent elements of $H^3(V_\Sigma - X)$ by forms $\frac{P\Omega_0}{f^k}$, where P is a polynomial in $\mathbb{C}[z_1, \ldots, z_q]$.

Mavlyutov described two induced residue maps on semiample hypersurfaces:

- $\mathrm{Res}_J : \mathbb{C}[z_1, \ldots, z_q]/J \to H^2(X)$ is well-defined for quasismooth hypersurfaces
- $\mathrm{Res}_{J_1} : \mathbb{C}[z_1, \ldots, z_q]/J_1 \to H^2(X)$ is well-defined for regular hypersurfaces.

Whither injectivity?

Res_J is injective for smooth hypersurfaces in $\mathbb{P}^3$, but this does not hold in general.

Theorem

[M00] If X is a regular, semiample hypersurface, then the residue map Res_{J_1} is injective.

The Griffiths-Dwork technique

Plan

We want to compute the Picard-Fuchs equation for a one-parameter family of K3 hypersurfaces X_t.

- Look for $\mathbb{C}(t)$-linear relationships between derivatives of periods of the holomorphic form
- Use Res_J to convert to a polynomial algebra problem in $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$

The Griffiths-Dwork technique

Procedure

1.

$$\frac{d}{dt}\int \mathrm{Res}\left(\frac{P\Omega}{f^k(t)}\right) = \int \mathrm{Res}\left(\frac{d}{dt}\left(\frac{P\Omega}{f^k(t)}\right)\right)$$
$$= -k\int \mathrm{Res}\left(\frac{f'(t)P\Omega}{f^{k+1}(t)}\right)$$

The Griffiths-Dwork technique

Procedure

1. $$\frac{d}{dt}\int \operatorname{Res}\left(\frac{P\Omega}{f^k(t)}\right) = \int \operatorname{Res}\left(\frac{d}{dt}\left(\frac{P\Omega}{f^k(t)}\right)\right)$$
$$= -k\int \operatorname{Res}\left(\frac{f'(t)P\Omega}{f^{k+1}(t)}\right)$$

2. Since $H^*(X_t, \mathbb{C})$ is a finite-dimensional vector space, only finitely many of the classes $\operatorname{Res}\left(\frac{d^j}{dt^j}\left(\frac{\Omega}{f^k(t)}\right)\right)$ can be linearly independent

The Griffiths-Dwork technique

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3. Use the reduction of pole order formula to compare classes of the form $\operatorname{Res}\left(\frac{P\Omega}{f^{k+1}(t)}\right)$ to classes of the form $\operatorname{Res}\left(\frac{Q\Omega}{f^k(t)}\right)$

The Griffiths-Dwork technique

Implementation

Reduction of pole order

$$\frac{\Omega_0}{f^{k+1}} \sum_i P_i \frac{\partial f}{\partial x_i} = \frac{1}{k} \frac{\Omega_0}{f^k} \sum_i \frac{\partial P_i}{\partial x_i} + \text{exact terms}$$

We use Groebner basis techniques to rewrite polynomials in terms of $J(f)$.

The Griffiths-Dwork technique

Advantages and disadvantages

Advantages

We can work with arbitrary polynomial parametrizations of hypersurfaces.

Disadvantages

We need powerful computer algebra systems to work with $J(f)$ and $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$.

Modular Groups and Modular Curves

- Consider a modular group $\Gamma \subset PSL_2(\mathbb{R})$.
- Γ acts on the upper half-plane $\mathbb{H}$ by linear fractional transformations:
$$z \mapsto \frac{az+b}{cz+d}$$
- $\overline{\mathbb{H}/\Gamma}$ is a Riemann surface called a modular curve.
- The function field of a genus 0 modular curve is generated by a transcendental function called a hauptmodul.

Some modular groups

Congruence subgroups

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \,\middle|\, c \cong 0 \pmod{n} \right\}$$

Atkin-Lehner map

$$w_h = \begin{pmatrix} 0 & \frac{-1}{\sqrt{h}} \\ \sqrt{h} & 0 \end{pmatrix} \in PSL_2(\mathbb{R})$$

$\Gamma_0(n) + h$ is generated by $\Gamma_0(n)$ and w_h.

Mirror Moonshine

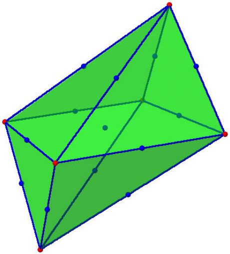
Mirror Moonshine for a one-parameter family of K3 surfaces arises when there exists a genus 0 modular group Γ such that . . .

- The Picard-Fuchs equation gives the base of the family the structure of a modular curve $\overline{\mathbb{H}/\Gamma}$, or a finite cover of the modular curve.
- The hauptmodul for Γ can be expressed as a rational function of the mirror map.
- The holomorphic solution to the Picard-Fuchs equation is a Γ-modular form of weight 2.

Mirror Moonshine from geometry

Example	[HLOY04]	[V96]
Shioda-Inose structure	$SI(E_1 \times E_2)$ $E_1,\ E_2$ are 6-isogenous	$SI(E_1 \times E_2)$ $E_1,\ E_2$ are 3-isogenous
$\mathrm{Pic}(X)^\perp$	$H \oplus \langle 12 \rangle$	$H \oplus \langle 6 \rangle$
Γ	$\Gamma_0(6) + 6$	$\Gamma_0(6) + 3 \subset \Gamma_0(3) + 3$

Geometry of the skew octahedron family



- X_t is a family of Kummer surfaces
- Each surface can be realized as $Km(E_t \times E_t)$
- The generic transcendental lattice is $2H \oplus \langle 4 \rangle$

The modular group

We use our symmetric square root and the table of [LW06] to show that:

$$\begin{aligned}\Gamma &= \Gamma_0(4|2) \\ &= \left\{ \left(\begin{array}{cc} a & b/2 \\ 4c & d \end{array} \right) \in PSL_2(\mathbb{R}) \;\middle|\; a, b, c, d \in \mathbb{Z} \right\}\end{aligned}$$

$\Gamma_0(4|2)$ is conjugate in $PSL_2(\mathbb{R})$ to $\Gamma_0(2) \subset PSL_2(\mathbb{Z}) = \Gamma_0(1) + 1$.

Batyrev, V. and Cox, D. On the Hodge structure of projective hypersurfaces in toric varieties. *Duke Mathematical Journal* 75, 1994.

Doran, C. Picard-Fuchs uniformization and modularity of the mirror map. *Communications in Mathematical Physics* 212 (2000), no. 3, 625–647.

Hosono, S., Lian, B.H., Oguiso, K., and Yau, S.-T. Autoequivalences of derived category of a $K3$ surface and monodromy transformations. *Journal of Algebraic Geometry* 13, no. 3, 2004.

Karp, D., Lewis, J., Moore, D., Skjorshammer, D., and Whitcher, U. "On a family of K3 surfaces with $\mathcal{S}_4$ symmetry". *Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds*, Fields Institute Communications.

- Lian, B.H. and Wiczer, J.L. Genus Zero Modular Functions, 2006.
  http://people.brandeis.edu/~lian/Schiff.pdf
- Mavlyutov, A. Semiample hypersurfaces in toric varieties. *Duke Mathematical Journal* 101 (2000), no. 1, 85–116.
- Narumiya, N. and Shiga, H. The mirror map for a family of K3 surfaces induced from the simplest 3-dimensional reflexive polytope. *Proceedings on Moonshine and related topics*, AMS 2001.
- Nikulin, V. Finite automorphism groups of Kähler K3 surfaces. *Transactions of the Moscow Mathematical Society* 38, 1980.
- SAGE Mathematics Software, Version 3.4,
  http://www.sagemath.org/

Verrill, H. Root lattices and pencils of varieties. *Journal of Mathematics of Kyoto University* 36, no. 2, 1996.