Modelling electricity futures by ambit fields

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Aim of the project

Modelling electricity futures

Futures contract

A futures contract maturing at time T on an asset S is a traded asset with "price" $F(t,T)$ such that the futures contract can be entered at zero cost at any time; a holder of the contract receives payments corresponding to the price changes of $F(t,T)$. At maturity T, $F(T,T) = S_T$.

Let t denote the current time and T the time of maturity/delivery. How can we model the futures/forward price $F(t,T)$?

- **Spot-based approach**: Let S denote the underlying spot price. Then

$$F(t,T) = \mathbb{E}^Q(S_T|\mathcal{F}_t).$$

- **Reduced-form modelling**: As in the Heath-Jarrow-Morton (HJM) framework, one can model $F(t,T)$ directly.

Stylised facts of electricity futures

- Non-Gaussian, (semi-) heavy-tailed distribution
- Volatility clusters and time-varying volatility
- Strong seasonality (over short and long time horizons)
- Presence of the “Samuelson effect”: Volatility of the futures contract increases as time to delivery approaches.
- Electricity is essentially not storable ➠ spikes, negative prices in the spot
- High degree of idiosyncratic risk ➠ use random fields!

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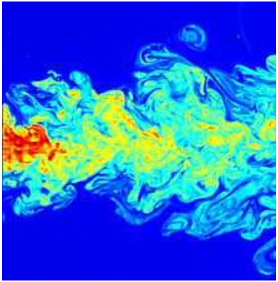
Use ambit fields to model electricity futures!

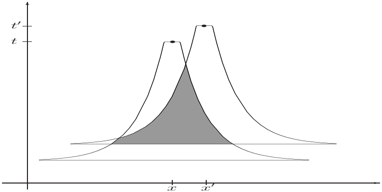
Ambit stochastics

- Name for the theory and applications of ambit fields and ambit processes
- Probabilistic framework for tempo-spatial modelling
- Introduced by O. E. Barndorff-Nielsen and J. Schmiegel in the context of modelling **turbulence** in physics.


What is an ambit field?

- **Aim**: Model real-valued tempo-spatial object $Y_t(x)$, where $t \in \mathbb{R}$ is the temporal and $x \in \mathbb{R}^d$ the spatial variable ($d \in \mathbb{N}$).
- "**ambit**" from Latin *ambire* or *ambitus*: border, boundary, sphere of influence etc.
- Define **ambit set** $A_t(x)$: Intuitively: **causality cone**.



- **Ambit fields:** Stochastic integrals with respect to an independently scattered, infinitely divisible random measure L:

$$Y_t(x) = \int_{A_t(x)} h(x, t; \xi, s)\sigma(\xi, s)L(d\xi, ds)$$

- Integration in the L^2-sense as described in Walsh (1986).

The integrator L is chosen to be a Lévy basis

- Notation: $\mathcal{B}(\mathbb{R})$ Borel sets of $\mathbb{R}$; $\mathcal{B}_b(S)$ bounded Borel sets of $S \in \mathcal{B}(\mathbb{R})$.

Definition 1

A family $\{L(A) : A \in \mathcal{B}_b(S)\}$ of random variables in $\mathbb{R}$ is called an $\mathbb{R}$-valued **Lévy basis** on S if the following three properties hold:

1. The law of $L(A)$ is infinitely divisible for all $A \in \mathcal{B}_b(S)$.
2. If $A_1, \ldots, A_n$ are disjoint subsets in $\mathcal{B}_b(S)$, then $L(A_1), \ldots, L(A_n)$ are independent.
3. If $A_1, A_2, \ldots$ are disjoint subsets in $\mathcal{B}_b(S)$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}_b(S)$, then $L\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} L(A_i)\,, a.s.$, where the convergence on the right hand side is $a.s.$.

- Conditions (2)&(3) define an **independently scattered random measure**.

Cumulant function

- The a cumulant function of a *homogeneous* Lévy bases is given by

$$\mathrm{C}\left(\zeta, L(A)\right) = \mathrm{Log}(\mathbb{E}(\exp(i\zeta L(A)))$$
$$= \left[i\zeta a - \frac{1}{2}\zeta^2 b + \int_{\mathbb{R}} \left(e^{i\zeta z} - 1 - i\zeta z \mathbb{I}_{[-1,1]}(z) \right) \nu(dz) \right] leb(A),$$

  where $leb(\cdot)$ denotes the Lebesgue measure, and where $a \in \mathbb{R}$, $b \geq 0$ and ν is a Lévy measure on $\mathbb{R}$.
  [The logarithm above should be understood as the distinguished logarithm, see e.g. Sato (1999).]

- The characteristic quadruplet associated with L is given by (a, b, ν, leb).
- We call an infinitely divisible random variable L' with characteristic triplet given by (a, b, ν) the *Lévy seed* associated with L.
- Note: $L((0, t]) = L_t$ is a a Lévy process (for a hom. Lévy basis).

The model

- Consider a market with finite time horizon $[0, T^*]$ for some $T^* \in (0, \infty)$.
- Need to account for a *delivery period*: Model the futures price at time $t \geq 0$ with delivery period $[T_1, T_2]$ for $t \leq T_1 \leq T_2 \leq T^*$ say.
- Model the futures price with delivery period $[T_1, T_2]$ by

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F(t, T) dT, \tag{1}$$

where $F(t, T)$ is the *instantaneous futures price*.

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Model definition under risk neutral probability measure

Under the assumptions (**A**.1) - (**A**.7):

$$F(t, T) = \Lambda(T) + \int_{A_t} k(T; \xi, s) \sigma(\xi, s) L(d\xi, ds). \tag{2}$$

Musiela parametrisation with $x = T - t$ and $f_t(x) = F(t, x + t)$:

$$f_t(x) = \Lambda(t + x) + \int_{A_t} k(x + t; \xi, s) \sigma(\xi, s) L(d\xi, ds). \tag{3}$$

Model assumptions

A.1 L is a a homogeneous, square-integrable Lévy basis on $\mathbb{R}^2$, which has zero mean; its characteristic quadruplet is denoted by (a, b, ν, leb).

A.2 The filtration $\{\mathcal{F}_t\}_{t\in[-T^*,T^*]}$ is initially defined by $\mathcal{F}_t = \cap_{n=1}^{\infty}\mathcal{F}_{t+1/n}^0$, where $\mathcal{F}_t^0 = \sigma\{L(A,s) : A \in \mathcal{B}_b([0,T^*]), -T^* \leq s \leq t\}$, which is right-continuous by construction and then enlarged using the *natural enlargement*.

A.3 The positive random field $\sigma = \sigma(\xi, s) : \Omega \times \mathbb{R}^2 \to (0, \infty)$ denotes the so-called *stochastic volatility field* and is assumed to be independent of the Lévy basis L.

A.4 The function $k : [0, T^*] \times [0, T^*] \times [-T^*, T^*] \to [0, \infty)$ denotes the so-called *weight function*;

A.5 For each $T \in [0, T^*]$, the random field $(k(T;\xi,s)\sigma(\xi,s))_{(\xi,s)\in[0,T^*]\times[-T^*,T^*]}$ is assumed to be predictable and to satisfy the following integrability condition:

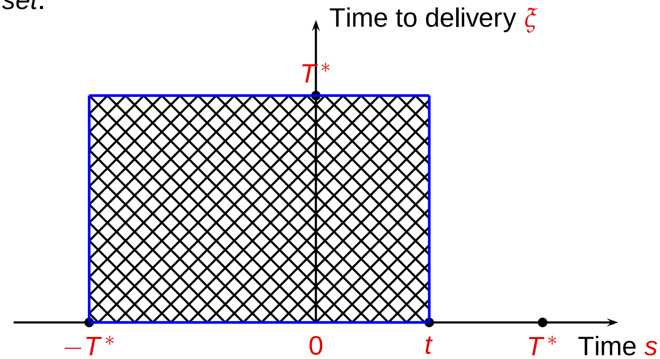
$$\mathbb{E}\left[\int_{[-T^*,T^*]\times[0,T^*]} k^2(T;\xi,s)\sigma^2(\xi,s)\, d\xi ds\right] < \infty. \tag{4}$$

Model assumptions cont'd

A.6 We call the set

$$A_t = [0, T^*] \times [-T^*, t] = \{(\xi, s) : 0 \leq \xi \leq T^*, -T^* \leq s \leq t\} \subseteq [0, T^*] \times [-T^*, T^*] \quad (5)$$

the *ambit set*.



A.7 The deterministic integrable function $\Lambda : [0, T^*] \to (0, \infty)$ denotes a seasonality and trend function.

Recap: The model

Let $0 \leq t \leq T \leq T^*$. Under the assumptions (**A**.1) - (**A**.7) the futures price under the risk-neutral probability measure is defined as the ambit field given by

$$F(t,T) = \Lambda(T) + \int_{A_t} k(T;\xi,s)\sigma(\xi,s)L(d\xi,ds). \tag{6}$$

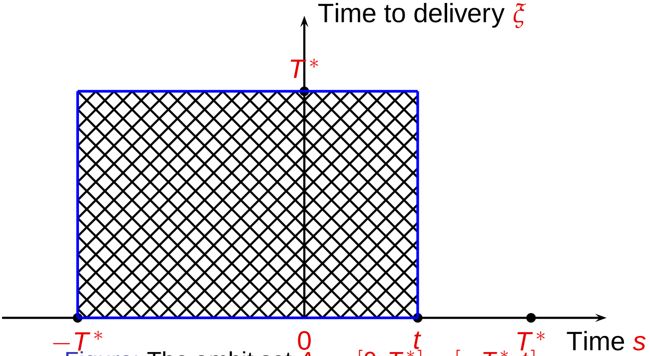


Figure: The ambit set $A_t = [0,T^*] \times [-T^*,t]$.

Important properties of the model

Proposition 2

For $T \in [0, T^*]$, the stochastic process $(F(t,T))_{0 \le t \le T}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$.

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Proposition 3

For $\mathcal{G}_t = \sigma\{\sigma(\xi, s), (\xi, s) \in A_t\}$, the conditional cumulant function we have

$$\mathrm{C}^{\sigma}(\zeta, f_t(x)) := \mathrm{Log}\left(\mathbb{E}\left(\exp(i\zeta f_t(x)) \middle| \mathcal{G}_t\right)\right)$$
$$= i\zeta\Lambda(t + x) + \int_{A_t} \mathrm{C}\left(\zeta k(x + t; \xi, s)\, \sigma(\xi, s), L'\right) \mathrm{d}\xi \mathrm{d}s,$$

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where L' is the Lévy seed associated with L.

Example 4

If L is a homogeneous Gaussian Lévy basis, then we have

$$\mathrm{C}\left(\zeta k\left(x+t;\xi,s\right)\sigma\left(\xi,s\right), L'\right) = i\zeta\Lambda(t+x) - \frac{1}{2}\zeta^2 k^2\left(x+t;\xi,s\right)\sigma^2\left(\xi,s\right).$$

Correlation structure

- Note that our new model does not only model one particular futures contract, but it models the entire futures curve at once.
- Let $0 \leq t \leq t+h \leq T^*$ and $0 \leq x, x' \leq T^*$, then

$$
\begin{aligned}
&Cor\left(f_t(x), f_{t+h}(x')\right) \\
&\qquad = K^{-1} \int_{A_t} k(x+t, \xi, s) k(x'+t+h, \xi, s) \mathbb{E}\left(\sigma^2(\xi, s)\right) d\xi ds,
\end{aligned}
$$

where

$$
\begin{aligned}
K = &\sqrt{\int_{A_t} k^2(x+t, \xi, s) \mathbb{E}\left(\sigma^2(\xi, s)\right) d\xi ds} \\
&\qquad \cdot \sqrt{\int_{A_t} k^2(x'+t+h, \xi, s) \mathbb{E}\left(\sigma^2(\xi, s)\right) d\xi ds}
\end{aligned}
$$

Examples of weight functions

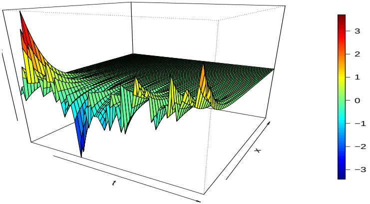
- Consider weight functions which factorise as

$$k(x+t;\xi,s) = \Phi(\xi)\Psi(x+t,s), \tag{7}$$

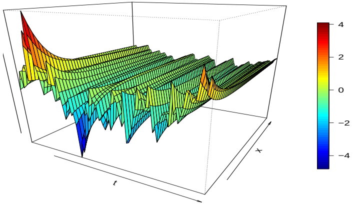
  for suitable functions Ψ and Φ. [In the case that $\Phi \equiv 1$ and there is no stochastic volatility we essentially get be back the classical framework.]

- OU-type weight function: $\Psi(x+t,s) = \exp(-\alpha(x+t-s))$, for some $\alpha > 0$.
- CARMA-type weight function: $\Psi(x+t-s) = \mathbf{b}' \exp(\mathbf{A}(x+t-s))\mathbf{e}_p$;
- Bjerksund et al. (2010)-type weight function: $\Psi(x+t,s) = \frac{a}{x+t-s+b}$, for $a, b > 0$
- Audet et al. (2004)-type weight function:
  - $\Psi(x+t,s) = \exp(-\alpha(x+t-s))$ for $\alpha > 0$,
  - $\Phi(\xi) = \exp(-\beta\xi)$, for $\beta > 0$

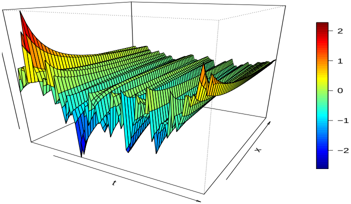
Example: Gaussian ambit fields



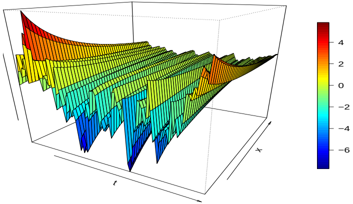
(a) Exponential weight function



(b) Sum of two exponential weight functions



(c) Bjerksund et al.-type weight function



(d) Gamma-type weight function

Implied spot price

- By the no-arbitrage assumption, the futures price for a contract which matures in zero time, $x = 0$, has to be equal to the spot price, that is, $f_t(0) = S_t$. Thus,

$$S_t = \Lambda(t) + \int_{A_t} k(t; \xi, s)\sigma(\xi, s)L(d\xi, ds).$$

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$$S_t = \Lambda(t) + \int_{A_t} k(t; \xi, s)\sigma(\xi, s)L(d\xi, ds).$$

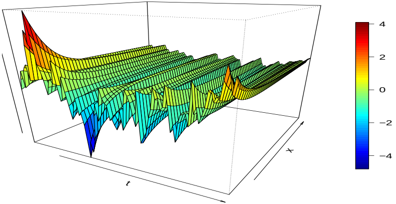
- In the Gaussian case, we get the following result:

$$S_t \stackrel{law}{=} \Lambda(t) + \int_{-T^*}^{t} \Psi(t; s)\omega_s dW_s,$$

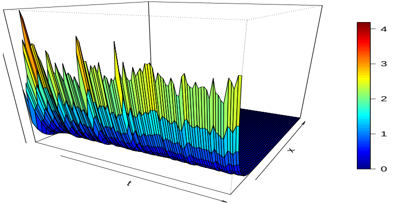
assuming that $k(x + t; \xi, s) = \Phi(\xi)\Psi(x + t, s)$, $\omega_s^2 = \int_0^{T^*} \Phi^2(\xi)\sigma^2(\xi, s)d\xi$ and where W is a Brownian motion.

- Null-spatial case of ambit field: Volatility modulated Volterra process, Lévy semistationary process. (Fit energy spot prices very well!)

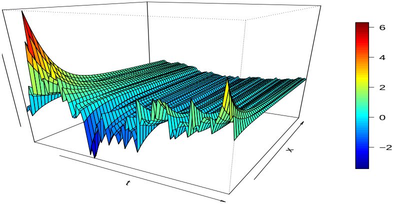
Simulated futures curve



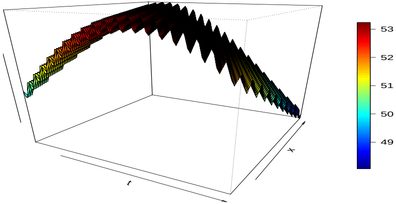
(e) Ambit field without stochastic volatility



(f) Stochastic volatility field

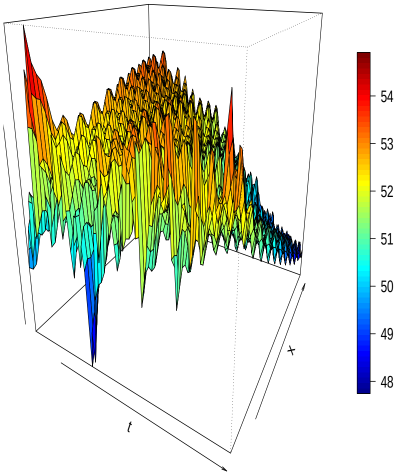


(g) Ambit field with stochastic volatility

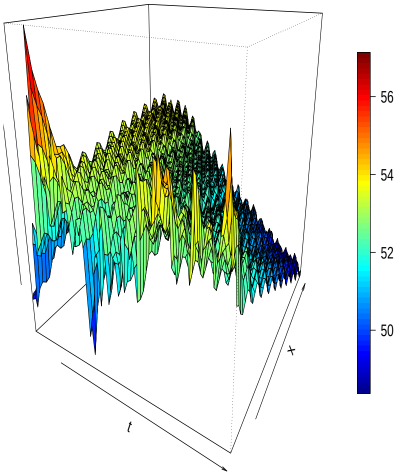


(h) Seasonality field

Simulated futures curve cont'd



(i) Futures price without stochastic volatility



(j) Futures price with stochastic volatility

Samuelson effect

- Samuelson effect: The volatility of the futures price increases when the time to delivery approaches zero.
- Also, the volatility of the futures converges to the volatility of the spot price.
- The weight function k plays the role of a damping function and is therefore non-increasing in the first variable and ensures that the Samuelson effect can be accounted for in our model.

Proposition 5

Under suitable conditions (given in our paper) the variance of the futures price $f_t(x)$, given by

$$v_t(x) := Var(f_t(x)) = c \int_{A_t} k^2(x+t; \xi, s)\mathbb{E}\left(\sigma^2(\xi, s)\right) d\xi ds,$$

is monotonically non-decreasing as $x \downarrow 0$. Further, the variance of the futures converges to the variance of the implied spot price.

Samuelson effect: Example for different choices of the weight function

Example 6

Suppose the weight function factorises as mentioned before and there is no stochastic volatility. Then the variance of the futures price is given by

$$\upsilon_t(x) = c' \int_{-T^*}^{t} \Psi^2(x+t,s)ds, \qquad \text{where } c' = c \int_0^{T^*} \Phi^2(\xi)d\xi.$$

This implies that in the context an exponential weight, we get

$$v_t(x) = c' \frac{1}{2\alpha} \left(e^{-2\alpha x} - e^{-2\alpha(x+t+T^*)} \right),$$

and in the context of the Bjerksund et al. (2010) model we have

$$v_t(x) = c' a^2 \left(\frac{1}{x+b} - \frac{1}{x+t+T^*+b} \right).$$

Change of measure

- Next we do a change of measure from the risk-neutral pricing measure to the physical measure.

Proposition 7

Define the process

$$M_t^\theta = \exp\left(\int_{A_t} \theta(\xi, s)\, L(d\xi, ds) - \int_{A_t} C(-i\theta(\xi, s), L')\, d\xi\, ds\right). \quad (8)$$

The deterministic function $\theta : [0, T^*] \times [-T^*, T^*] \mapsto \mathbb{R}$ *is supposed to be integrable with respect to the Lévy basis* L *in the sense of Walsh (1976). Assume that*

$$\mathbb{E}\left(\exp\left(\int_{A_t} C(-i\theta(s, \xi), L')\, d\xi\, ds\right)\right) < \infty, \text{ for all } t \in \mathbb{R}_{T^*}. \quad (9)$$

Then M_t^θ *is a martingale with respect to* $\mathcal{F}_t$ *with* $\mathbb{E}[M_0^\theta] = 1$.

Change of measure cont'd

- Define an equivalent probability P by

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} = M_t^{\theta}\,, \tag{10}$$

for $t \geq 0$, where the function θ is an additional parameter to be modelled and estimated, which plays the role as the *market price of risk*

- We compute the characteristic exponent of an integral of L under P.

Proposition 8

For any $v \in \mathbb{R}$, and Walsh-integrable function g with respect to L, it holds that

$$\mathrm{Log}\mathbb{E}_P\left[\exp\left(iv\int_{A_t} g(\xi,s)L(d\xi,ds)\right)\right] = \int_{A_t}\left(C(vg(\xi,s)-i\theta(\xi,s),L') - C(-i\theta(\xi,s),L')\right)\,d\xi\,ds\,.$$

Summary of key results

- Use ambit fields to model electricity futures.
- Our model ensures that the futures price is a martingale under the risk-neutral measure.
- Studied relevant examples of model specifications.
- New modelling framework accounts for the key stylised facts observed in electricity futures.
- Futures and spot prices can be linked to each other within the ambit field framework (Samuelson effect).
- Change of measure.

Further results not mentioned today:

- Geometric modelling framework
- Option pricing based on Fourier techniques.
- Simulation methods for ambit fields.

Outlook

- Detailed empirical studies.
- Inference methods for ambit fields.
- Need for more efficient simulation schemes.

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