Partitions and Quantum Groups

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Compact Matrix Quantum Groups & Easy QG's

One possible motivation for quantum groups (C^*-alg. approach)
Symmetries of a (compact) space X: group $(G,\circ)$ acting on X
dualization: $(\mathcal{C}(G),\Delta)$ coacting on $\mathcal{C}(X)$
noncommutative version: quantum group coacting on a C^*-algebra

Definition (Woronowicz 87)

Let $n\in\mathbb{N}$. A *compact matrix quantum group* consists of

- a unital C^*-algebra A
- generated by elements u_{ij}, $1\leq i,j\leq n$ (*-algebra is dense)
- such that $u=(u_{ij})$ and $u^t=(u_{ji})$ are invertible
- and a *-homomorphism $\Delta: A\to A\otimes_{\min} A$, $u_{ij}\mapsto\sum_k u_{ik}\otimes u_{kj}$

Remark: comp. matrix QG $\Rightarrow$ compact QG, Haar state exists.

Ex. 1: Orthogonal group $O_n \subseteq M_n(\mathbb{C})$
$\rightsquigarrow$ *free orthogonal quantum group* O_n^+ [Wang 95]

$$\mathcal{C}(O_n) = C^*(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^*, uu^t = u^t u = 1, u_{ij}u_{kl} = u_{kl}u_{ij})$$
$$\mathcal{C}(O_n^+) := A_o(n) := C^*(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^*, uu^t = u^t u = 1)$$

Note: matrix multiplication $\circ : O_n \times O_n \to O_n$
translates to comultiplication $\Delta : \mathcal{C}(O_n) \to \mathcal{C}(O_n) \otimes \mathcal{C}(O_n)$

Ex. 2: permutation group $S_n \subseteq M_n(\mathbb{C})$
$\rightsquigarrow$ *quantum permutation group* S_n^+ [Wang 98]

$$\mathcal{C}(S_n) = C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1, u_{ij}u_{kl} = u_{kl}u_{ij})$$
$$\mathcal{C}(S_n^+) := A_s(n) := C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1)$$

Woronowicz' Tannaka-Krein result [Woronowicz 88]:
Study *intertwiner space* of a compact matrix QG G ($\forall k, l \in \mathbb{N}_0$).

$$\mathrm{Hom}_G(k,l) = \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \text{ linear} \mid Tu^{\otimes k} = u^{\otimes l} T\}$$

Idea of **indexing the maps T by partitions** [Brauer 37].
Let $p \in P(k,l)$ be a partition on k upper and l lower points.

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) := n^{-\frac{1}{2}\beta(p)} \sum_{j_1,\ldots,j_l} \delta_p(i,j) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

We have:

$$\begin{aligned}
\mathrm{Hom}_{O_n}(k,l) &= \mathrm{span}\{T_p \mid p \in P_{\mathrm{pair}}(k,l)\} \\
\mathrm{Hom}_{O_n^+}(k,l) &= \mathrm{span}\{T_p \mid p \in NC_{\mathrm{pair}}(k,l)\} \\
\mathrm{Hom}_{S_n}(k,l) &= \mathrm{span}\{T_p \mid p \in P(k,l)\} \\
\mathrm{Hom}_{S_n^+}(k,l) &= \mathrm{span}\{T_p \mid p \in NC(k,l)\}
\end{aligned}$$

Definition (Banica, Speicher 09)

A compact matrix quantum group $S_n \subseteq G \subseteq O_n^+$ is called *easy* (or: *partition quantum group*), if

$$\mathrm{Hom}_G(k,l) = \mathrm{span}\{T_p \mid p \in \mathcal{C}(k,l)\}, \qquad \text{for all } k,l \in \mathbb{N}_0$$

for a collection $\mathcal{C}$ of subsets $\mathcal{C}(k,l) \subseteq P(k,l)$, $k,l \in \mathbb{N}_0$.

The set $\mathcal{C}$ is a *category of partitions* (since Hom_G is a tensor cat.):

- $p,q \in \mathcal{C} \Rightarrow p \otimes q \in \mathcal{C}$ (horizontal concat., $T_p \otimes T_q = T_{p\otimes q}$)
- $p,q \in \mathcal{C} \Rightarrow pq \in \mathcal{C}$ (vertical concat., $T_p T_q = n^{-\gamma(p,q)} T_{pq}$)
- $p \in \mathcal{C} \Rightarrow p^* \in \mathcal{C}$ (upside-down, $(T_p)^* = T_{p^*}$)
- $\sqcap \in P(0,2)$ and $| \in P(1,1)$ are in $\mathcal{C}$

Why are easy quantum groups interesting (in free prob.)?

- Their **combinatorics** is given by partitions (all: group case, noncrossing: free case; but: more than P vs. NC)
- Give rise to appropriate symmetries (**de Finetti** theorems etc. [Köstler, Speicher, Banica, Curran])
- **envelopping von Neumann algebras** are somehow related to $L\mathbb{F}_n$ ($G = O_n^+, U_n^+$ [Vaes, Vergnioux, Banica, Brannan, Freslon, Isono,...])
- **stochastic aspects** (Diaconis-Shahshahani type results, distributions of characters etc. [Banica, Curran, Speicher, Belinschi, Capitaine, Collins,...])

Classification of easy QG's

- $\exists!$ 7 free easy QG's (categories noncrossing) [Banica, Speicher 09, W. 13; (Banica, Bichon, Collins 07)]
- $\exists!$ 6 easy groups (categ. containing  $\in P(2,2)$, $u_{ij}u_{kl} = u_{kl}u_{ij}$) [Banica, Speicher 09]
- $\exists!$ 3 half-liberated easy QG's & one infinite series (categories containing  $\in P(3,3)$, $u_{ij}u_{kl}u_{st} = u_{st}u_{kl}u_{ij}$) [Banica, Curran, Speicher 10, W. 13]
- $\exists!$ 13 *non-hyperoctahedral* easy QG's ($\sim$ categories containing singletons as blocks) [Banica, Curran, Speicher 10, W. 13]
- *hyperoctahedral* case: [Raum, W. 12 & 13]

Details on the hyperoctahedral case (joint with Sven Raum)

Let: G easy QG, $\mathcal{C}$ associated categ. of partitions, no singletons.
Case 1. [Raum, W. 12] $\sqcup\!\!\!/\!\sqcap \in \mathcal{C}$, i.e. $u_{ij}u_{kl}^2 = u_{kl}^2 u_{ij}$.
Obtain a **group structure** out of $\mathcal{C}$:

- Label the blocks of the partitions $p \in \mathcal{C}$ by letters $a_1, a_2, \ldots$
- Need only those p whith mutually different neighbouring letters
- Obtain a subgroup $F(\mathcal{C}) \subseteq \mathbb{Z}_2^{*\infty}$, invariant under certain endomorphism actions ($gh \simeq p \otimes q$, $g^{-1} \simeq p^*$, end. actions $\simeq$ pq and others)
- Yields lattice isomorphism F between class of such categories of partitions and certain subgroups of $\mathbb{Z}_2^{*\infty}$
$\Rightarrow$ Classification of easy QG $=$ Classif. of those subgroups (huge class)

- In particular: Lattice injection of the class of non-empty varieties of groups (uncountably many) into easy QG's
  $\Rightarrow$ easy QG's form a rich class!
- Link to quantum isometry groups/ symmetric reflection groups: Let H be an "appropriate" subgroup of $\mathbb{Z}_2^{*\infty}$ and consider $C^*_{\max}(\mathbb{Z}_2^{*n}/(H)_n)$. The maximal quantum subgroup of $H_n^{[\infty]}$ (corresponding to the category generated by $\sqcup\!\!\!/\!\sqcap$) acting faithfully and isometrically on this C^*-algebra is exactly imposed by the category $F^{-1}(H)$.

Case 2. [Raum, W. 13, coming soon] $\sqcup\!\!\!/\!\sqcap \notin \mathcal{C}$, but $\times\!\!\!\times \in \mathcal{C}$, i.e. $u_{ij}^2 u_{kl}^2 = u_{kl}^2 u_{ij}^2$. (Almost) purely combinatoric.

Fusion rules of easy QG's (joint with Amaury Freslon)

Repr.: $(u^\alpha_{st}) \in M_{n_\alpha}(\mathbb{C}) \otimes \mathcal{C}(G)$ unitary s.t. $\Delta(u^\alpha_{st}) = \sum_r u^\alpha_{sr} \otimes u^\alpha_{rt}$
Woronowicz: (u^α_{st}) decomposes into a direct sum of irr. rep.'s
u^α, u^β irr. rep's $\Rightarrow$ $u^\alpha \otimes u^\beta = \sum_\gamma u^\gamma$ *fusion rules* (= "group law")

Fusion rules for S_n^+ and O_n^+ are known [Banica 90's], but we can now treat all easy QG's uniformly, using partitions!
[Freslon, W. 13, coming soon]

$p = p^* = pp \in \mathcal{C}(k,k)$ *projective* partition $\Rightarrow$ T_p projection
$u_p := (\mathrm{id} \otimes P_p) u^{\otimes k}$, where $P_p := T_p - \bigvee T_q$ and $q \in \mathcal{C}(k,k)$ runs through all projective partitions s.t. $pq = q \neq p$ (i.e. $q < p$)

In general, u_p is *not* irreducible, but $\mathrm{Aut}(u_p) \cong \bigoplus_{\alpha \in J(p)} M_{n_\alpha}(\mathbb{C})$, where $\mathbb{C}[\mathrm{Sym}_{\mathcal{C}}(p)] = \bigoplus_{\alpha \in J(p) \cup I(p)} M_{n_\alpha}(\mathbb{C})$, and $\mathrm{Sym}_{\mathcal{C}}(p) \subseteq S_m$.
S_n^+, O_n^+: $\mathrm{Sym}_{\mathcal{C}}(p)$ is trivial, hence u_p irred.

$u_p \otimes u_q = \sum_m u_m$, where m runs through all partitions $p *_h q \in \mathcal{C}$
S_n^+, O_n^+: partitions $h \in NC$ only of two/one kind

u_p, u_q unitarily equivalent iff $p = r^* r, q = rr^*$ for some $r \in \mathcal{C}$
iff $\#\{\text{through-blocks}(p)\} = \#\{\text{thr.-blocks}(q)\} \Rightarrow$ indexed by $\mathbb{N}_0$

S_n^+: $u_k \otimes u_l = u_{|k-l|} \oplus u_{|k-l|+1} \oplus \ldots \oplus u_{k+l-1} \oplus u_{k+l}$
O_n^+: $u_k \otimes u_l = u_{|k-l|} \oplus u_{|k-l|+2} \oplus \ldots \oplus u_{k+l-2} \oplus u_{k+l}$

Summary/ work in progress

- [W. 13]: On the classification of free (noncrossing partitions), half-liberated ($u_{ij}u_{kl}u_{st} = u_{st}u_{kl}u_{ij}$) and non-hyperoctahedral ("containing singletons") easy QG's
- [Raum, W. 12]: classification in the hyperoctahedral case I ($u_{ij}u_{kl}^2 = u_{kl}^2 u_{ij}$, subgroups of $\mathbb{Z}_2^{*\infty}$)
- [Raum, W. 13, coming soon]: hyperoctahedral case II ($u_{ij}^2 u_{kl}^2 = u_{kl}^2 u_{ij}^2$), completing the classification
- [Freslon, W. 13, coming soon]: Fusion rules for all easy QG, using partitions
- [Tarrago, W., work in progress]: Unitary easy QG ($u_{ij} \neq u_{ij}^*$, using colored partitions)