Block modifications of the Wishart ensemble and operator-valued free multiplicative convolution

Carlos Vargas Obieta

joint work with

S. Belinschi, R. Speicher, J. Treilhard (arXiv:1209.3508)

O. Arizmendi, I. Nechita (on going)

Universität des Saarlandes, Saarbrücken

Toronto, July 4th, 2013

(some) Previous work

1986 Voiculescu: Addition of free random variables.
$G_x \rightsquigarrow \phi_x$
$\phi_x(z) + \phi_y(z) = \phi_{x+y}(z) \rightsquigarrow G_{x+y}(z)$

1991 - : First results on asymptotic freeness
1995 - : Operator-valued random variables
1996 Shlyakthenko: Band matrices
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$X_1^{(N)}, X_2^{(N)}, \ldots, X_p^{(N)}$ Independent Gaussian (or H. U.) matrices
$N \to \infty$
$\to s_1, s_2, \ldots, s_p$ Free semicircular random variables

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Replace the state

$$\tau : \mathcal{M} \to \mathbb{C}$$

by a (unit preserving) conditional expectation

$$\mathbb{E} : \mathcal{M} \to \mathcal{B} \supseteq \mathbb{C}$$

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$(\mathcal{M}, \tau_N)$ $\mathcal{M}$: random $N \times N$ matrices.

$$\begin{array}{rcl} \mathbb{E} : M_2(\mathcal{M}) & \to & M_2(\mathbb{C}) \\ \begin{pmatrix} X_1 & X_2 \\ X_2 & X_1 \end{pmatrix} & \mapsto & \begin{pmatrix} \tau_N(X_1) & \tau_N(X_2) \\ \tau_N(X_2) & \tau_N(X_1) \end{pmatrix} \\ & \to & \begin{pmatrix} s_1 & s_2 \\ s_2 & s_1 \end{pmatrix} \end{array}$$

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$\mathbb{E} : \mathcal{M} \to < p_1, \ldots, p_k >$

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$$W^{\varphi} := (id_d \otimes \varphi)(W).$$

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φ "planar" $W^{\varphi} \to$ free compound Poissons (Banica, Nechita 2012).

Let us write

$$\varphi(A) = \sum_{i,j,k,l=1}^{n} \alpha_{kl}^{ij} E_{ij} A E_{kl},$$

where $E_{ij} \in \mathcal{M}_n(\mathbb{C})$ are matrix units.

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$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad \tau(e_{ij}) = n^{-1}\delta_{ij}, \quad \sum_{i=1}^n e_{ii} = 1, \quad e_{ij}^* = e_{ji}.$$

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Note: The (e_{ij}) are not free among themselves!

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so that

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From the elements f_{kl}^{ij}, $(i,j) \leq (l,k)$ we build a vector $f = (f_{11}^{11}, f_{12}^{11}, \ldots, f_{nn}^{nn})$ of size $m := n^2(n^2+1)/2$.

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We consider also the diagonal matrix $\tilde{w} = diag(\varepsilon_{11}^{11} w, \varepsilon_{12}^{11} w, \ldots, \varepsilon_{nn}^{nn} w)$, so that $f \tilde{w} f^* = w^\varphi$.

The desired distribution is the same (modulo a dirac mass at zero of weight $1 - 1/m$) as the distribution of $f^*f\tilde{w}$ in the C^*-probability space $(M_m(\mathbb{C}) \otimes \mathcal{A}, tr_m \otimes \tau)$.

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Moreover, since w and (e_{kl}^{ij}) are free, the matrices $f^* f$ and $\tilde{w}$ are free with amalgamation over $M_m(\mathbb{C})$ (with respect to the conditional expectation $\mathbb{E} := id_m \otimes \tau$).

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If we manage to compute matrix-valued free multiplicative convolutions, we would obtain the distribution of w^{φ} for ALL self-adjoint maps.

Free multiplicative convolution

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2006 Dykema : Operator valued S-Transform.

2012 (pre-print) Belinschi, Speicher, Treilhard, V.: Iterative analytic map approach to OVFMC

Operator-Valued free probability

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An *operator-valued non-commutative probability space* is a triple $(\mathcal{M}, \mathbb{E}, B)$,where $\mathcal{M}$ is a C^*-algebra, $B \subseteq \mathcal{M}$ is a C^*-subalgebra containing the unit of $\mathcal{M}$, and $\mathbb{E} \colon \mathcal{M} \to B$ is a unit-preserving conditional expectation. $(\mathbb{E}(bab') = b\mathbb{E}(a)b')$

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If $(\mathcal{A}, \tau)$ is a (scalar) C^* probability space, we may consider $(\mathcal{M}, \mathbb{E}, B)$

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Examples

If $(\mathcal{A}, \tau)$ is a (scalar) C^* probability space, we may consider $(\mathcal{M}, \mathbb{E}, B)$, where $\mathcal{M} = M_2(\mathbb{C}) \otimes \mathcal{A}$, $B = M_2(\mathbb{C})$ and

$$\begin{array}{rcl} \mathbb{E} : \mathcal{M} & \to & M_2(\mathbb{C}) \\ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} & \mapsto & \begin{pmatrix} \tau(a_1) & \tau(a_2) \\ \tau(a_3) & \tau(a_4) \end{pmatrix} \end{array}$$

or maybe $B = D_2(\mathbb{C})$ and

$$\begin{array}{rcl} \mathbb{E} : \mathcal{M} & \to & D_2(\mathbb{C}) \\ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} & \mapsto & \begin{pmatrix} \tau(a_1) & 0 \\ 0 & \tau(a_4) \end{pmatrix} \end{array}$$

B-Freeness

Definition

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Two algebras $A_1, A_2 \subseteq \mathcal{M}$ containing B are called *free with amalgamation over B* with respect to $\mathbb{E}$ (or just *free over B*) if for any tuple $x_1, \dots x_n$, such that $x_j \in A_{i_j}$ and $i_j \neq i_{j+1}$

$$\mathbb{E}[\bar{x}_1 \bar{x}_2 \cdots \bar{x}_n] = 0$$

where $\bar{x} := x - \mathbb{E}(x)$

Operator-valued Cauchy Transform

A very powerful tool for the study of operator-valued distributions is the generalized Cauchy-Stieltjes transform:

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If we also have scalar valued structure $(\mathcal{M}, \mathbb{E}, B)$, such that $\tau \circ \mathbb{E} = \tau$, then

$$\tau((z1_B - x)^{-1}) = \tau(\mathbb{E}((z1_B - x)^{-1}))$$

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are free with amalgamation over $M_n(\mathbb{C})$ (w.r.t $id_n \otimes \tau$).

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Main Definitions and Transforms

We shall use the following analytic mappings, all defined on $\mathbb{H}^+(B)$. in all formulas below, $x = x^*$ is fixed in $\mathcal{M}$:

The reciprocal Cauchy transform:

$$F_x(b) = \mathbb{E}\left[(b - x)^{-1}\right]^{-1} = G_x(b)^{-1}; \tag{2}$$

The eta transform (Boolean cumulant series):

$$\eta_x(b) = 1 - b\mathbb{E}\left[(b^{-1} - x)^{-1}\right]^{-1} = 1 - bF_x(b^{-1}); \tag{3}$$

We use an auxiliary "h transform:"

$$h_x(b) = b^{-1}\eta_x(b) = b^{-1} - \mathbb{E}\left[(b^{-1} - x)^{-1}\right]^{-1} = b^{-1} - F_x(b^{-1}); \tag{4}$$

Main Theorems

Theorem (Belinschi, Speicher, Treilhard, V. 2012)

Let $x > 0, y = y^* \in \mathcal{M}$ be two random variables with invertible expectations, free over B. There exists a Fréchet holomorphic map $\omega_2\colon \{b \in B\colon \Im(bx) > 0\} \to \mathbb{H}^+(B)$, such that

1. $\eta_y(\omega_2(b)) = \eta_{xy}(b)$, $\Im(bx) > 0$;
2. $\omega_2(b)$ *and* $b^{-1}\omega_2(b)$ *are analytic around zero;*
3. *For any* $b \in B$ *so that* $\Im(bx) > 0$, *the map* $g_b\colon \mathbb{H}^+(B) \to \mathbb{H}^+(B)$, $g_b(w) = bh_x(h_y(w)b)$ *is well-defined, analytic and*
$$\omega_2(b) = \lim_{n\to\infty} g_b^{\circ n}(w),$$
for any fixed $w \in \mathbb{H}^+(B)$.

Moreover, if one defines $\omega_1(b) := h_y(\omega_2(b))b$, *then*

$$\eta_{xy}(b) = \omega_2(b)\eta_x(\omega_1(b))\omega_2^{-1}(b), \quad \Im(bx) > 0.$$

Main Theorems

Theorem (Belinschi, Speicher, Treilhard, V. 2012)

Let B be finite-dimensional. For any $x \geq 0$, $y = y^*$ free over B, there exists a domain $\mathcal{D} \subset B$ containing $\mathbb{C}^+ \cdot 1$ and an analytic map $\omega_2 \colon \mathcal{D} \to \mathbb{H}^+(B)$ so that

$$\eta_y(\omega_2(b)) = \eta_{xy}(b) \text{ and } g_b(\omega_2(b)) = \omega_2(b), \quad b \in \mathcal{D}.$$

Moreover, if $g_b \colon \mathbb{H}^+(B) \to \mathbb{H}^+(B)$, $g_b(w) = bh_x(h_y(w)b)$, then $\omega_2(b) = \lim_{n\to\infty} g_b^{\circ n}(w)$, for any $w \in \mathbb{H}^+(B)$, $b \in \mathcal{D}$.

Example: product of (shifted) operator valued semicirculars

Let s_1, s_2, s_3, and s_4 be free, semi-circular random variables, in some scalar-valued non-commutative probability space $(\mathcal{A}, \tau)$. Consider the matrices S_1 and S_2 defined by:

$$S_1 = \begin{pmatrix} s_1 & s_1 \\ s_1 & s_2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} s_3 + s_4 & 2s_4 \\ 2s_4 & s_3 - 3s_4 \end{pmatrix} \tag{5}$$

Matrices S_1 and S_2 represent limits of random matrices, where $s_1, \ldots, s_4$ are replaced by independent Gaussian random matrices.

Numerical Implementation: Op-val Semicirculars

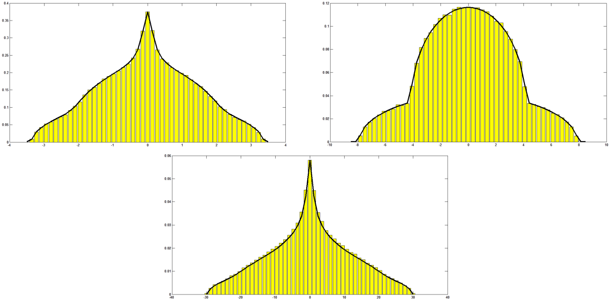

Figure: Spectral distributions of S_1 (left), S_2 (right), and $(S_2 + 8.5I_2)^{1/2} S_1 (S_2 + 8.5I_2)^{1/2}$ (center)

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we can compute the distribution of w^{φ}
for ALL self-adjoint maps.

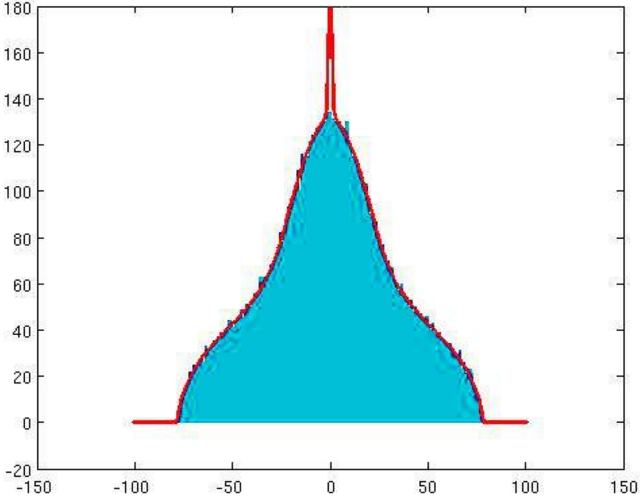

Figure: Block-modified Wigner matrix

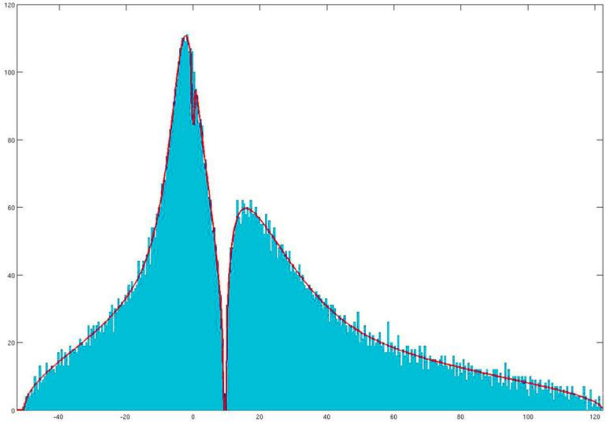

Figure: Block-modified Wishart matrix

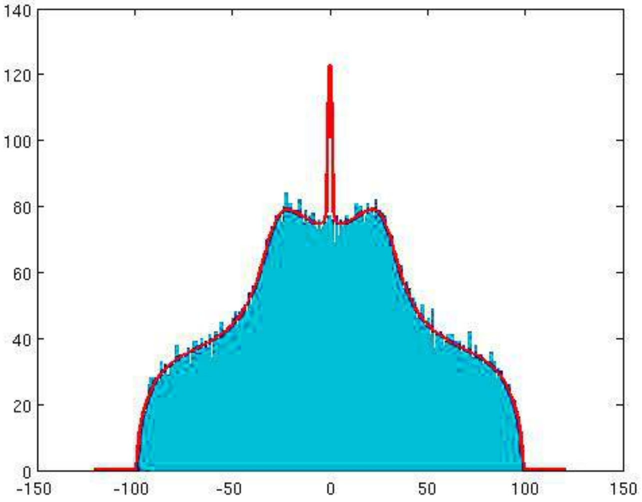

Figure: Block-modification of a rotated arcsine matrix

Reminder: Soccer at 3:00 pm

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Thanks for your attention!