Free Probability with Left and Right Variables

(Free Probability for Pairs of Faces)

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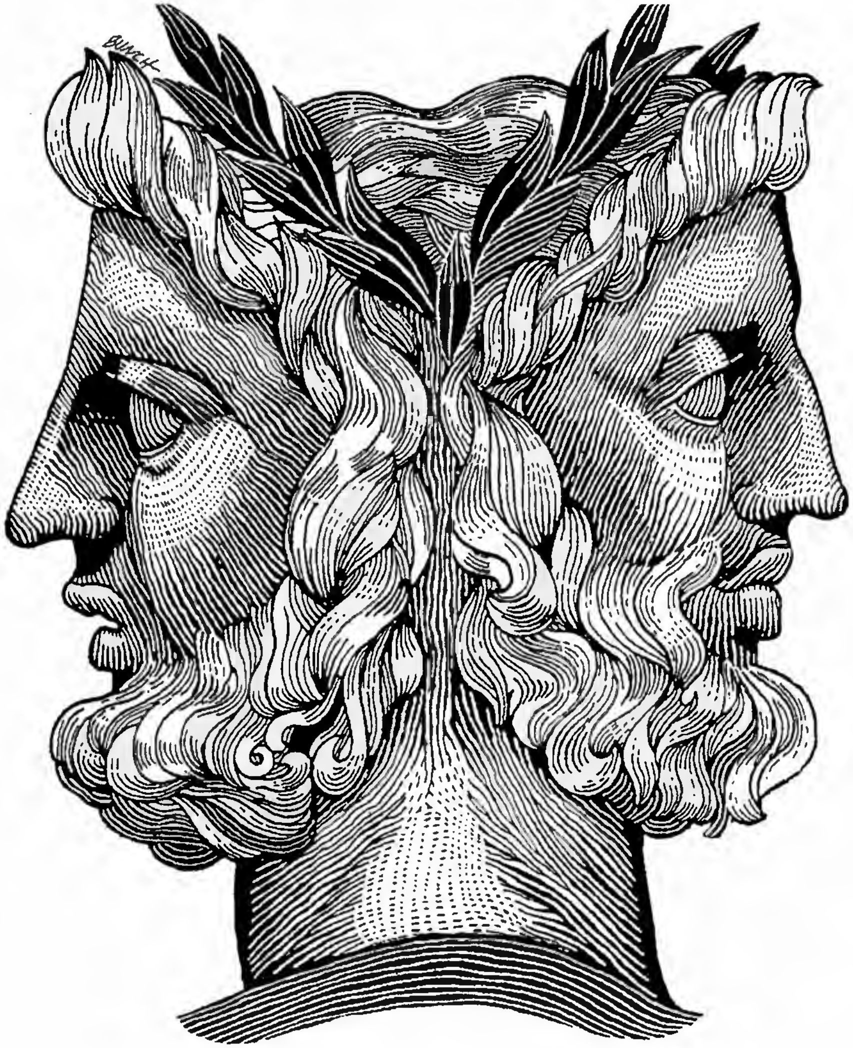

Janus

2 faces

Past and Future

Transition

[Left Var, Right Var] = 0

Bipartite System

Possible Connections:

- Free Probability of Type B
(Biane-Goodman-Nica, Belinschi-Shlyakhtenko)

- Second Order Freeness
(Collins-Mingo-Sniady-Speicher)

- Matricial Freeness
(Lenczewski)

Free Product of Vector Spaces with specified State Vectors

$$\mathcal{X}_\iota = \mathring{\mathcal{X}}_\iota \oplus \mathbb{C}\xi_\iota$$

$$\mathcal{X} = \mathbb{C}\xi \oplus \underbrace{\bigoplus_{n \geq 1} \bigotimes_{\iota_1 \neq \cdots \neq \iota_n} \mathring{\mathcal{X}}_{\iota_1} \otimes \cdots \otimes \mathring{\mathcal{X}}_{\iota_n}}_{\mathring{\mathcal{X}}}$$

$$(\mathcal{X}, \mathring{\mathcal{X}}, \xi) = \mathop{\ast}_{\iota \in I} (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota, \xi_\iota)$$

$$\varphi_\xi : \mathcal{L}(\mathcal{X}) \to \mathbb{C}, \quad T\xi \in \varphi_\xi(T)\xi \oplus \mathring{\mathcal{X}}.$$

Left and Right Factorizations

$$V_i : \mathcal{X}_i \otimes \Big(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigotimes_{i\neq i_1 \neq \cdots \neq i_n} \mathring{\mathcal{X}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{X}}_{i_n}\Big) \to \mathcal{X}$$

$$W_i : \Big(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigotimes_{i_1\neq \cdots \neq i_n \neq i} \mathring{\mathcal{X}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{X}}_{i_n}\Big) \otimes \mathcal{X}_i \to \mathcal{X}$$

$$T \in \mathcal{L}(\mathcal{X}_i)$$

$$\lambda_i(T) = V_i (T \otimes I) V_i^{-1} \in \mathcal{L}(\mathcal{X})$$

$$\rho_i(T) = W_i (I \otimes T) W_i^{-1} \in \mathcal{L}(\mathcal{X})$$

$$[\lambda_i(T), \rho_j(S)] = \delta_{ij} [T, S] \oplus 0 .$$

$(\mathcal{A}, \varphi)$ noncommutative probability space

Pair of Faces in $(\mathcal{A}, \varphi)$

$(\mathcal{B}, \beta)$ left face, right face $(\mathcal{C}, \gamma)$

$$\begin{tikzcd} & \mathcal{A} & \\ \mathcal{B} \arrow[ur, "\beta"] & & \mathcal{C} \arrow[ul, "\gamma"'] \end{tikzcd}$$

β, γ unital homomorphisms

$\mathcal{B}, \mathcal{C}$ unital algebras

Included faces $\mathcal{B} \subset \mathcal{A} \supset \mathcal{C}$.
(β, γ are the inclusions)

2-faced family of noncommutative random variables in $(\mathcal{A}, \varphi)$

$$\left((b_i)_{i \in I}, (c_j)_{j \in J}\right) \text{ in } \mathcal{A}$$

[Corresponds to

$$\beta : \mathbb{C}\langle X_i \mid i \in I \rangle \longrightarrow \mathcal{A}, \quad \beta(X_i) = b_i$$

$$\gamma : \mathbb{C}\langle Y_j \mid j \in J \rangle \longrightarrow \mathcal{A}, \quad \gamma(Y_j) = c_j]$$

Bi-freeness of a family of pairs of faces $((B_\iota, \beta_\iota), (C_\iota, \gamma_\iota))_{\iota \in I}$ in (A, φ):

$\exists\ (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota, \xi_\iota)$, $\iota \in I$, $(\mathcal{X}, \mathring{\mathcal{X}}, \xi) = \mathop{*}\limits_{\iota \in I} (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota, \xi_\iota)$

$l_\iota : B_\iota \longrightarrow \mathcal{L}(\mathcal{X}_\iota)$, $r_\iota : C_\iota \longrightarrow \mathcal{L}(\mathcal{X}_\iota)$ unital homomorphisms, so that

$$\varphi \circ \pi = \varphi_\xi \circ \tilde{\pi}$$

$\pi : \mathop{*}\limits_{\iota \in I} (B_\iota * C_\iota) \longrightarrow A$, $\pi|B_\iota = \beta_\iota$, $\pi|C_\iota = \gamma_\iota$

$\tilde{\pi} : \mathop{*}\limits_{\iota \in I} (B_\iota * C_\iota) \longrightarrow \mathcal{L}(\mathcal{X})$, $\pi|B_\iota = \lambda_\iota \circ l_\iota$, $\pi|C_\iota = \rho_\iota \circ r_\iota$.

<u>Remarks</u>: 1° If $((B_\iota, \beta_\iota), (C_\iota, \gamma_\iota))_{\iota \in I}$ bi-free in $(\mathcal{A}, \varphi)$, joint distribution $\varphi \circ \tilde{\pi}$ obtained also as $\varphi_{\xi'} \circ \tilde{\pi}'$ for any other $(\mathcal{X}'_\iota, \mathring{\mathcal{X}}'_\iota, \xi'_\iota)$, ℓ'_ι, r'_ι so that

$$\varphi \circ \tilde{\pi}_\iota = \varphi_{\xi'_\iota} \circ \tilde{\pi}'_\iota$$

$$\tilde{\pi}_\iota : B_\iota * C_\iota \longrightarrow \mathcal{A}, \quad \tilde{\pi}_\iota|_{B_\iota} = \beta_\iota, \quad \tilde{\pi}_\iota|_{C_\iota} = \gamma_\iota$$

$$\tilde{\pi}'_\iota : B_\iota * C_\iota \longrightarrow \mathcal{L}(\mathcal{X}'_\iota), \quad \tilde{\pi}'_\iota|_{B_\iota} = \ell'_\iota, \quad \tilde{\pi}'_\iota|_{C_\iota} = r'_\iota$$

2°. $\left((B_\iota,\beta_\iota),(C_\iota,\gamma_\iota)\right)_{\iota\in I}$ bi-free in $(\mathcal{A},\varphi)$

then $(\beta_\iota(B_\iota))_{\iota\in I}$ free in $(\mathcal{A},\varphi)$

$\iota\neq j \Rightarrow \beta_\iota(B_\iota),\ \gamma_j(C_j)$ classically independent in $(\mathcal{A},\varphi)$.

3°. bi-freeness has the necessary properties to be used as an independence relation in a noncommutative probability theory with left- and right variables i.e. two-faced families.

4°. C^*-bi-freeness, W^*-bi-freeness
bi-free products of states etc.
bi-free convolution operations
(additive, multiplicative)

$\mu \boxplus\boxplus \nu$, $\mu \boxtimes\boxtimes \nu$

Bi-freeness Examples

I. Groups $(G_\iota)_{\iota \in I}$, $G = \ast_{\iota \in I} G_\iota$.

$L_\iota : \mathbb{C}[G_\iota] \to \mathcal{L}(\mathbb{C}[G])$

$R_\iota : \mathbb{C}[G_\iota]^{op} \to \mathcal{L}(\mathbb{C}[G])$

restrictions of left and right regular representations

$((\mathbb{C}[G_\iota], L_\iota), (\mathbb{C}[G_\iota]^{op}, R_\iota))_{\iota \in I}$

bi-free family of faces in $(\mathbb{C}[G], \tau)$.

v. Neumann trace

II. Left and right creation and annihilation operators on the full Fock space.

$\mathcal{H}$ complex Hilbert sp. $(e_\iota)_{\iota\in I}$ ONB

$$\mathcal{T}(\mathcal{H}) = \mathbb{C}1 \oplus \bigoplus_{n\geq 1} \mathcal{H}^{\otimes n}$$

$$l_\iota \xi = e_\iota \otimes \xi\,,\quad r_\iota \xi = \xi \otimes e_\iota\,,\quad \xi \in \mathcal{T}(\mathcal{H}).$$

$\omega(T) = \langle T1, 1\rangle$ on $\mathcal{B}(\mathcal{T}(\mathcal{H}))$

$\left((l_\iota, l_\iota^*), (r_\iota, r_\iota^*)\right)_{\iota\in I}$

bi-free in $\left(\mathcal{B}(\mathcal{T}(\mathcal{H})), \omega\right)$.

Bi-free Cumulants

$z = \left((z_i)_{i \in I}, (z_j)_{j \in J} \right)$ 2-faced family of r.v. in (A, φ)

Moments $\varphi(z_{\alpha(1)} \cdots z_{\alpha(n)})$, $\alpha : \{1, \dots, n\} \to I \amalg J$

R_α polynomial in commuting variables $X_{\alpha(k_1) \cdots \alpha(k_r)}$, $1 \le k_1 < \dots < k_r \le n$.

homogeneous deg $= n$, $\deg X_{\alpha(k_1) \cdots \alpha(k_r)} = r$.

$$R_\alpha(z) = R_\alpha(\varphi(z_{\alpha(k_1)} \cdots z_{\alpha(k_r)}) \mid 1 \le k_1 < \dots < k_r \le n \}$$

R_α bi-free cumulant, exists & unique so that:

1°. coefficient of $X_{\alpha(1)\dots\alpha(n)} = 1$

2°. z', z'' bi-free in (A, φ), then

$$R_\alpha(z') + R_\alpha(z'') = R_\alpha(z' + z'').$$

$$\alpha\Pi_n = \{(\alpha(k_1), \dots \alpha(k_r)) \mid 1 \le k_1 < \dots < k_r \le n,\ 1 \le r \le n\}$$

$$M_{z,\alpha} = \left(\varphi(z_{\alpha(1)} \cdots z_{\alpha(k_r)})\right)_{(\alpha(k_1), \dots \alpha(k_r)) \in \alpha\Pi_n}$$

$$(M_{z',\alpha}, M_{z'',\alpha}) \longrightarrow M_{z'+z'',\alpha}$$

polynomial abelian group law on $\mathbb{C}^{\alpha\Pi_n}$

$$\underset{(\text{Lie algebra}, +)}{\mathbb{C}^{\alpha\Pi_n}} \xrightarrow{\exp} \underset{⊞⊞_n \text{ law}}{\mathbb{C}^{\alpha\Pi_n}}$$ isomorphi

$\log = (\exp)^{-1}$ yields cumulants

Bi-free Central Limit

z two-faced family in (A, φ)

has bi-free central limit distribution

(aka bi-free Gaussian)

if $n \neq 2 \Longrightarrow R_{\alpha(1)\cdots\alpha(n)}(z) = 0$

$n = 1$ $\quad R_a(z) = \varphi(z_a)$

$n = 2$ $\quad R_{ab}(z) = \varphi(z_a z_b) - \varphi(z_a)\varphi(z_b)$.

bi-free central limit distribution

$\gamma_C : \mathbb{C}\langle Z_k \mid k \in I \amalg J \rangle \longrightarrow \mathbb{C}$

determined by covariance matrix

$C = (C_{k,\ell})_{k,\ell \in I \amalg J}$.

$\gamma_C(Z_k Z_\ell) = C_{k\ell}$

(equivalently $C_{k\ell} = R_{k\ell}(Z)$).

Realization on full Fock space

$\mathcal{T}(\mathcal{H})$ full Fock space, $T \underset{\text{vacuum expectation}}{\longrightarrow} \langle T1, 1\rangle$

$\ell(h), \ell^*(h), r(h), r^*(h)$
left and right creation and annihilation

$h, h^* : I \amalg J \longrightarrow \mathcal{H}$ maps

$z_i = \ell(h(i)) + \ell^*(h^*(i)) \quad i \in I$

$z_j = r(h(j)) + r^*(h^*(j)) \quad j \in J$

$z = ((z_i)_{i\in I}, (z_j)_{j \in J})$ bi-free Gaussian

covariance $C_{ab} = \langle h(b), h^*(a)\rangle$.

Bi-free Algebraic CLT

bi-free sequence

$$(z^{(n)})_{n\in\mathbb{N}} = \left(\left((z_i^{(n)})_{i\in I}, (z_j^{(n)})_{j\in J} \right) \right)_{n\in\mathbb{N}} \text{ in } (A,\varphi)$$

(i) $\varphi(z_k^{(n)}) = 0$, $k \in I \sqcup J$

(ii) $\sup\limits_{n\in\mathbb{N}} |\varphi(z_{k_1}^{(n)} \cdots z_{k_m}^{(n)})| = D_{k_1 \dots k_m} < \infty$

(iii) $\lim\limits_{N\to\infty} N^{-1} \sum\limits_{1\le n\le N} \varphi(z_k^{(n)} z_\ell^{(n)}) = C_{k\ell}$

$$S_N = \left((S_{N,i})_{i\in I}, (S_{N,j})_{j\in J} \right)$$

$$S_{N,k} = N^{-1/2} \sum_{1\le n\le N} z_k^{(n)}$$

$\Rightarrow S_N$ has limit distribution bi-free Gaussian with covariance $(C_{k\ell})_{k,\ell\in I\sqcup J}$ as $N \to \infty$

$\mathbb{C} \rightsquigarrow B$ algebra with 1.

Bi-freeness with amalgamation over B

<u>B-B noncommutative probability space</u>

$(\mathcal{A}, p, \varepsilon)$ $\mathcal{A}$ unital algebra over $\mathbb{C}$

$\varepsilon : B \otimes B^{op} \longrightarrow \mathcal{A}$ unital homomorphism

$\varepsilon|_{B \otimes 1}$, $\varepsilon|_{1 \otimes B^{op}}$ injective

$p : \mathcal{A} \longrightarrow B$ linear unital

$p(\varepsilon(b_1 \otimes 1)\, a\, \varepsilon(1 \otimes b_2)) = b_1\, p(a)\, b_2$

(in particular $(p \circ \varepsilon)(b_1 \otimes b_2) = b_1 b_2$.

$(\mathcal{A}, p, \varepsilon)$ B-B noncommutative probability space

$\mathcal{A}_r$ commutant in $\mathcal{A}$ of $\varepsilon(1 \otimes B^{op})$

$\mathcal{A}_\ell$ commutant in $\mathcal{A}$ of $\varepsilon(B \otimes 1)$

included pair of B-faces in $(\mathcal{A}, p, \varepsilon)$

(C, D) unital subalgebras in $\mathcal{A}$

$$\varepsilon(B \otimes 1) \subset C \subset \mathcal{A}_r$$

$$\varepsilon(1 \otimes B^{op}) \subset D \subset \mathcal{A}_\ell$$

B-B bimodules with specified state vector

$\mathcal{X} = \overset{\circ}{\mathcal{X}} \oplus B$ $\qquad$ $\mathcal{X}, \overset{\circ}{\mathcal{X}}$ B-B bimodules

Free Product

$$\mathop{\ast_B}_{\iota \in I} (\mathcal{X}_\iota, \overset{\circ}{\mathcal{X}}_\iota) = (\mathcal{X}, \overset{\circ}{\mathcal{X}})$$

$$\overset{\circ}{\mathcal{X}} = \bigoplus_{n \geq 1} \bigoplus_{\iota_1 \neq \cdots \neq \iota_n} \mathcal{X}_{\iota_1} \otimes_B \mathcal{X}_{\iota_2} \otimes_B \cdots \otimes_B \mathcal{X}_{\iota_n}$$

$$\mathcal{X} = \overset{\circ}{\mathcal{X}} \oplus B$$

$\mathcal{X} = \mathring{\mathcal{X}} \oplus B$ B-B bimodule

$p_{\mathcal{X}} : \mathcal{L}(\mathcal{X}) \longrightarrow B$

$T(0 \oplus 1) \in \mathring{\mathcal{X}} \oplus p_{\mathcal{X}}(T)$

$\varepsilon_{\mathcal{X}} : B \otimes B^{op} \longrightarrow \mathcal{L}(\mathcal{X})$ left & right B multiplication

$(\mathcal{L}(\mathcal{X}), p_{\mathcal{X}}, \varepsilon_{\mathcal{X}})$ B-B noncommutative probability space

$\mathcal{L}_r(\mathcal{X})$ right B-linear operators

$\mathcal{L}_l(\mathcal{X})$ left B-linear operators

$$(\mathcal{X}, \mathring{\mathcal{X}}) = \mathop{\ast}_{\iota \in I}{}_B (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota)$$

$$V_\iota : \mathcal{X}_\iota \otimes_B \Big(B \oplus \bigoplus_{m \geq 1} \bigoplus_{\iota \neq \iota_1 \neq \cdots \neq \iota_m} \mathring{\mathcal{X}}_{\iota_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{\iota_m}\Big) \to \mathcal{X}$$

$$W_\iota : \Big(B \oplus \bigoplus_{m \geq 1} \bigoplus_{\iota_1 \neq \cdots \neq \iota_m \neq \iota} \mathring{\mathcal{X}}_{\iota_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{\iota_m}\Big) \otimes \mathcal{X}_\iota \to \mathcal{X}$$

$$\lambda_\iota : \mathcal{L}_r(\mathcal{X}_\iota) \longrightarrow \mathcal{L}_r(\mathcal{X})$$

$$\rho_\iota : \mathcal{L}_\ell(\mathcal{X}_\iota) \longrightarrow \mathcal{L}_\ell(\mathcal{X})$$

$$\lambda_\iota(T) = V_\iota (T \otimes 1) V_\iota^{-1}$$

$$\rho_\iota(T) = W_\iota (I \otimes T) W_\iota^{-1}$$

$(\mathcal{A}, p, \varepsilon)$ B-B noncomm. prob. sp.

<u>family</u> $((C_\iota, D_\iota))_{\iota \in I}$ of <u>pairs of</u> B-<u>face</u> <u>in</u> $(\mathcal{A}, p, \varepsilon)$ <u>is bi-free over</u> $\mathcal{B}$ if:

$\exists \quad \mathcal{X}_\iota = \mathring{\mathcal{X}}_\iota \oplus B$ B-B bimodules

unital homomorphisms

$\gamma_\iota : C_\iota \longrightarrow \mathcal{L}_r(\mathcal{X}_\iota)$, $\gamma_\iota(\varepsilon(b \otimes 1)) = \varepsilon_{\mathcal{X}_\iota}(b \otimes 1)$

$\delta_\iota : D_\iota \longrightarrow \mathcal{L}_\ell(\mathcal{X}_\iota)$, $\delta_\iota(\varepsilon(1 \otimes b)) = \varepsilon_{\mathcal{X}_\iota}(1 \otimes b)$

so that

if $c_k \in C_{\iota(k)}$, $d_k \in D_{\iota(k)}$, $1 \le k \le n$

then

$$p(c_1 d_1 c_2 d_2 \cdots c_n d_n) =$$

$$= p_{\mathfrak{X}}\left(\lambda_{i(1)}(\gamma_{\iota(1)}(c_1))\, \rho_{\iota(1)}(\delta_{\iota(1)}(d_1)) \cdots \right.$$

$$\left. \cdots \lambda_{i(n)}(\gamma_{\iota(n)}(c_n))\, \rho_{\iota(n)}(\delta_{\iota(n)}(d_n))\right).$$