Introduction to Banach and Operator Algebras

Lecture 5

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Operators on Hilbert Spaces

Let H be a Hilbert space with inner product

$$\langle \xi \mid \eta \rangle$$

for $\xi, \eta \in H$. We obtain a norm

$$\|\xi\| = \langle \xi \mid \xi \rangle^{\frac{1}{2}}.$$

A linear operator $x : H \to H$ is bounded if

$$\|x\| = \sup\{\|x\xi\| : \|\xi\| \leq 1, \xi \in H\}.$$

Then $B(H)$, the space of bounded linear operators on H, is a Banach space.

Involution on $B(H)$

$B(H)$ with this operator norm is unital Banach algebra since

$$\|xy\| \leq \|x\|\|y\|.$$

There exist an involution $*$ on $B(H)$ given by

$$\langle x^*\xi \mid \eta\rangle = \langle \xi \mid x\eta\rangle.$$

$B(H)$ with this involution is an involutive Banach algebra since it satisfies

(1) $(\alpha x + \beta y)^* = \bar{\alpha}x^* + \bar{\beta}y^*$, (2) $(xy)^* = y^*x^*$, (3) $(x^*)^* = x$.

Moreover it also satisfies

(4) $\|x^*x\| = \|x\|^2$.

Therefore, $B(H)$ is a unital C*-algebra.

C*-algebras

In general, a C*-algebra is an involutiva Banach algebra satisfying the condition (4), i.e. it satisfies

$$\|x^*x\| = \|x\|^2.$$

It is clear that every norm closed *-subalgebra

$$A \subseteq B(H)$$

is a C*-algebra. Here we say that A is *-subalgebra if $x^* \in A$ whenever $x \in A$.

Theorem [Gelfand-Naimark 1943]: Let A be a C*-algebra, i.e. let A be an involutive Banach algebra satisfying the condition (4). Then there exists a Hilbert space H and an isometric *-homomorphism

$$\pi : A \to \pi(A) \subseteq B(H).$$

This shows that every C*-algebra can be represented on some Hilbert space.

Examples of C*-algebras

- $B(H)$ for some Hilbert space H.

In particular the matrix algebra $M_n(\mathbb{C}) = B(\mathbb{C}^n)$, for which the multiplication is given by matrix product

$$[x_{ij}][y_{jk}] = [\sum_j x_{ij}y_{jk}]$$

and the involution is given by $[x_{ij}]^* = [\bar{x}_{ji}]$

- Finite dimensional C*-algebras

$$M_{n_1}(\mathbb{C}) \oplus_\infty M_{n_2}(\mathbb{C}) \oplus_\infty \cdots \oplus_\infty M_{n_k}(\mathbb{C}).$$

- The space $K(H) \subseteq B(H)$ of all compact linear operators on H

- Any norm closed ideal J of a C*-algebra A, and its quotient A/J

- The Calkin algebra $Q(H) = B(H)/K(H)$

Commutative C*-algebras

Let Ω be a compact topological space. Then $A = C(\Omega)$ with norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in \Omega\}$$

and involution $f^*(t) = \overline{f(t)}$ is a unital commutative C*-algebra.

Indeed, for any $f, g \in C(\Omega)$, we have

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$$

and we have

$$\|f^* f\|_\infty = \sup\{|\overline{f(t)} f(t)| : t \in \Omega\} = \|f\|_\infty^2.$$

Therefore, $(C(\Omega), \|\cdot\|_\infty)$ is a unital commutative C*-algebra.

Theorem: For every unital commutative C*-algebra A, there exists a compact topological space Ω such that we have the isometric *-isomorphism

$$A = C(\Omega).$$

Proof: Let A be a unital commuative C*-algebra and let

$$\Omega = \Delta(A)$$

be the set of all unital *-homomorphism (i.e. unital contractive homomorphism) from A to $\mathbb{C}$. Then Ω is a weak* closed and thus weak* compact subset of A_1^*. Let

$$a \in A \to \hat{a} \in A^{**}$$

be the canonical isometric inclusion given by

$$\hat{a}(\varphi) = \varphi(a)$$

for $\varphi \in A^*$. Then the Gelfand Transformation

$$a \in A \to \hat{a}_{|\Omega} \in C(\Omega)$$

is an isometric *-isomorphism from A onto $C(\Omega)$, Remark:

Let Ω be a compact topological space. For each $t \in \Omega$, the point-evaluation

$$\varphi_t : f \in C(\Omega) \to f(t) \in \mathbb{C}$$

is a unital *-homomorphism from $C(\Omega)$ into $\mathbb{C}$. This defines a homeomorphism

$$\tau : t \in \Omega \leftrightarrow \varphi_t \in \Delta(C(\Omega)).$$

Therefore, the above Theorem establshes a duality correspondence between

Compact Topological Spaces Ω
and
Unital Comm C*-algebras $A = C(\Omega)$.

We also have a natural duality correspondence between

Locally Compact Topological Spaces Ω

and

Commutative C*-algebras $C_0(\Omega)$

Therefore, we may regard general

C*-algebras

as

Noncommutative Topological Spaces

More Exmaples of C*-algebras

Group C*-algebras $C^*_\lambda(G)$

Let G be a discrete group and $H = \ell_2(G)$. For each $s \in G$, we obtain a unitary operator λ_s on $\ell_2(G)$ given by

$$(\lambda_s \xi)(t) = \xi(s^{-1}t).$$

We have

$$\lambda_s \lambda_t = \lambda_{st} \text{ and } \lambda_s^* = \lambda_{s^{-1}}.$$

Then $C^*_\lambda(G) = \{\sum_{s\in G} \alpha_s \lambda_s\}^{-\|\cdot\|}$ is a unital C*-subalgebra of $B(\ell_2(G))$. We call $C^*_\lambda(G)$ the reduced group C*-algebra.

If G is an abelian group, then $C^*_\lambda(G)$ is a unital comm C*-algebra. In this case, each unital *-homomorphism $\varphi : C^*_\lambda(G) \to \mathbb{C}$ uniquely corresponds to a group homomorphism

$$\chi_\varphi : s \in G \to \varphi(\lambda_s) \in \mathbb{T} \subseteq \mathbb{C}.$$

In this case, $\Delta(C^*_\lambda(G))$ is just the dual group $\widehat{G} = \{\chi : G \to \mathbb{T}\}$ all (continuous) characters of G.

• If $G = \mathbb{Z}$, then $\widehat{G} = \mathbb{T}$ and thus

$$C^*_\lambda(\mathbb{Z}) = C(\mathbb{T}).$$

• If $G = \mathbb{Z} \times \mathbb{Z}$, then

$$C^*_\lambda(\mathbb{Z} \times \mathbb{Z}) = C(\mathbb{T} \times \mathbb{T}).$$

• If $G = \mathbb{F}_2$ is the free group of 2-generators, then $C^*_\lambda(\mathbb{F}_2)$ represents a noncomutative topological space.

Suppose that $\mathbb{F}_2$ is the free group with two generators u and v. Then $\mathbb{F}_2$ consists of all reduced words: e (empty word), u, v, u^{-1}, v^{-1} (words of length 1, $uu, uv, uv^{-1}, vv, vu, vu^{-1}, u^{-1}u^{-1}, \cdots$ (words of length 2), $\cdots$.

Question: How many elements of length $|s| = n$?

Then $\mathbb{F}_2$ is a non-abelian group with multiplication and inverse given by

$$(uvu^{-1})(uvvu) = uvvvu \text{ and } (uvu^{-1})^{-1} = uv^{-1}u^{-1}.$$

The empty word e is the unital element of $\mathbb{F}_2$.

Reduced Free Group C*-algebras

Theorem [Powers 1975]: $C^*_\lambda(\mathbb{F}_2)$ is a simple C*-algebra, i.e. has no non-trivial closed two-sided ideals.

Remark: The simplicity of $C^*_\lambda(\mathbb{F}_2)$ means that the corresponding "space" is highly noncommutative.

Theorem [Pimsner and Voiculescu 1982] and [Connes 1986]: $C^*_\lambda(\mathbb{F}_2)$ has no non-trivial projection.

Remark: If we have a non-trivial projection $p = \chi_E$ in $C(\Omega)$, then the corresponding set E must be closed and open in Ω. Therefore, Ω must be disconnected.

Therefore, the above theorem shows that $C^*_\lambda(\mathbb{F}_2)$ determines a

"highly noncommutative and connected space. "

Rotation Algebras

Let us first recall that we can identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ via the function $z(t) = e^{2\pi it}$. We let $H = L_2(\mathbb{T}) = L_2(\mathbb{R}/\mathbb{Z})$.

Let θ be a real number in $[0, 1)$. We can obtain two unitary operators U and V on H given by

$$U\xi(t) = z(t)\xi(t) \text{ and } V\xi(t) = \xi(t - \theta).$$

A simple calculation shows that

$$UV = e^{2\pi i\theta} VU.$$

Let A_θ be the universal C*-algebra generated by the unitary operators $\tilde{U}$ and $\tilde{V}$ satisfying the above relation. We call A_θ the rotation algebra.

If $\theta = 0$, we get $UV = VU$. In this case,

$$A_0 \cong C(\mathbb{T} \times \mathbb{T})$$

is a unital commutative C*-algebra.

We are particularly interested in the case when θ is irrational.

Theorem [Rieffel 1981]: If θ is an irrational number, then A_θ is a unital simple C*-algebra.

Since $VU = e^{-2\pi i\theta}UV$, we get $VU = e^{2\pi i(1-\theta)}UV$, and thus

$$A_\theta = A_{1-\theta}.$$

However, for distinct irrationals θ in $[0, \frac{1}{2}]$, A_θ are all distinct (i.e. non-isomorphic).

CAR Algebra

Let us consider the canonical embeddings

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes M_2 \otimes M_2 \hookrightarrow ...$$

Take the norm closure, we get a C*-algebra A_{2^∞}, which is called the CAR algebra.

If we consider all projections in the diagonal of A_{2^∞}. These projections generates a unital commutative C*-algebra $B = C(\Omega)$, where Ω is nothing, but the Cantor set.

von Neumann Algebras

Let H be a Hilbert space. We say that a net of operators $\{x_\alpha\}$ converges to x in the strong operator topology in $B(H)$ if

$$\|x_\alpha \xi - x\xi\| \to 0 \quad \text{for all } \xi \in H.$$

A von Neumann algebra on a Hilbert space H is a strong operator closed *-subalgebra $M \subseteq B(H)$. So every von Neumann algebra is a C*-algebra and is a dual space with a unique predual. In general speaking, von Neumann algebras are exactly dual C*-algebras.

Let (X, μ) be a measure space. Then $L_\infty(X, \mu)$ is a commutative von Neumann algebra on $L_2(X, \mu)$. In fact, every commutative von Neumann algebra M can be written as $M = L_\infty(X, \mu)$.

There is a correspondence between

Measure Spaces (X, μ)

and

Commutative von Neumann Algebras $L_\infty(X, \mu)$

Therefore, we may regard general

von Neumann Algebras

as

Noncommutative Measure Spaces

Examples

Let G be a discrete group. Then the group von Neumann algebra

$$VN_\lambda(G) = span\{\lambda_s : s \in G\}^{-s.o.t}.$$

is a von Neumann algebra.

If $G = \mathbb{Z}$, then $VN_\lambda(\mathbb{Z}) = L_\infty(\mathbb{T})$.

If $G = \mathbb{Z} \times \mathbb{Z}$, then $VN_\lambda(\mathbb{Z} \times \mathbb{Z}) = L_\infty(\mathbb{T} \times \mathbb{T})$.

In general, we may regard $VN_\lambda(G) \cong L_\infty(\widehat{G})$ as the duality of $L_\infty(G)$ Here $\widehat{G}$ is just a notation to indicate the 'duality' of G.

There exists a unique normal tracial state τ on $VN_\lambda(G)$ given by

$$\tau(x) = \langle x\delta_e | \delta_e \rangle$$

which corresponding to the canonical Haar measure on $\widehat{G}$.

Hyperfinite II_1-Factor

- Consider the canonical embeddings

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes M_2 \otimes M_2 \hookrightarrow \ldots$$

We may take a "weak closure" and obtain a von Neumann algebra R_{2^∞}.

- We can, similarly, consider the von Neumann algebra R_{3^∞} generated by 3×3 matrices.

It turns out that these von Neumann algebras are equal ! They are all hyperfinite II_1-factor.

A von Neumann algebra M on a Hilbert space H is called a factor if

$$M \cap M' = \mathbb{C}1,$$

where $M' = \{x \in B(H) : xy = yx, y \in M\}$ is the commutant of M. A von Neumann algebra is called hyperfinite if it contains sufficiently many finite dim C*-subalgebras.

Appendix I

Let G be a discrete group. Then $\ell_1(G)$ is a unital involutive Banach algebra with the multiplication given by the convolution

$$f \star g(t) = \sum_{s \in G} f(s) g(s^{-1} t)$$

and the involution given by

$$f^*(t) = \overline{f(t^{-1})}.$$

Let δ_s denote the characteristic function at s. Then for $s, t \in G$, we have

$$\delta_s \star \delta_t = \delta_{st}.$$

From this it is easy to see that δ_e is the unit element of $\ell_1(G)$.

Theorem: If $|G| \geq 2$, $\ell_1(G)$ is not a C*-algebra, i.e. it fails to have

$$\|f^* \star f\|_1 = \|f\|_1^2.$$

Example 1: We can look at $\ell_1(\mathbb{Z})$, and consider $f = \delta_0 + i\delta_1 + \delta_2$. It is easy to see that $\|f\|_1 = 3$. But

$$f^* \star f = (\delta_0 - i\delta_{-1} + \delta_{-2}) \star (\delta_0 + i\delta_1 + \delta_2) = \delta_{-2} + 3\delta_0 + \delta_2.$$

So

$$\|f^* \star f\|_1 = 5 < 9 = \|f\|_1^2.$$

Example 2: Find a function $f \in \ell_1(\mathbb{Z}_2)$ such that

$$\|f^* \star f\|_1 \neq \|f\|_1^2.$$

Appendix II

Let A be a C*-algebra. Then

$$A_{s.a} = \{a \in A : a^* = a\},$$

the space of all selfadjoint operators in A, is a real subspace of A.

An operator $a \in A$ is positive if a is selfadjoint and its spectrum $\sigma(a) \subseteq [0, \infty)$. An operator $a \in A$ is positive if and only if $a = b^*b$ for some $b \in A$. Then A^+, the set of all positive operators in A, is a proper positive cone in $A_{s.a.}$. This defines an order on $A_{s.a.}$, i.e. $a \leq b$ if $b - a \geq 0$.

Theorem: Every selfadjoint element $a \in A_{s.a.}$ can be uniquely decomposed to

$$a = a^+ - a^- \text{ with } a^+ a^- = 0.$$

Example: Let $A = C(\Omega)$. Then $A_{s.a.} = C(\Omega, \mathbb{R})$ and $A^+ = C(\Omega, [0, \infty))$.

GNS Representation

A linear functional $\varphi : A \to \mathbb{C}$ is positive if

$$\varphi : A^{+} \to [0, \infty).$$

Every positive linear functional is bounded with $\|\varphi\| = \varphi(1)$.

Theorem [Gelfand-Naimark-Segal]: Let $\varphi : A \to \mathbb{C}$ be a positive linear functional. There exist a Hilbert space H_φ, a unital *-homomorphism $\pi_\varphi : A \to B(H_\varphi)$, and a vector $\xi_\varphi \in H_\varphi$ such that

$$\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi | \xi_\varphi \rangle.$$

We can choose H_φ such that $\pi_\varphi(A)\xi_\varphi$ is norm dense in H_φ. In this case, we call $(\pi_\varphi, H_\varphi, \xi_\varphi)$ is a (cyclic) GNS representation of φ.

Outline of Proof: First, we can define a semi-inner product on A given by

$$\langle a|b\rangle_\varphi = \varphi(b^*a).$$

Let $N_\varphi = \{a \in A : \varphi(a^*a) = 0\}$. Then N_φ is a left ideal of A, and the above semi-inner product induces an inner product

$$\langle [a]|[b]\rangle_\varphi = \varphi(b^*a) \text{ for } [a], [b] \in A/N_\varphi.$$

We let H_φ denote the norm completion of A/N_φ.

For each $x \in A$, we can define a bounded operator

$$\pi_\varphi(x) : [a] \in A/N_\varphi \to [xa] \in A/N_\varphi$$

with $\|\pi_\varphi(x)\| \leq \|x\|$. We use $\pi_\varphi(x)$ denote the extension to H_φ. Then

$$\pi_\varphi : x \in A \to \pi_\varphi(x) \in B(H_\varphi).$$

is a unital *-homomorphism. Finally, we let $\xi_\varphi = [1] \in H_\varphi$ and get

$$\varphi(x) = \varphi(1^*x) = \langle [x]|[1]\rangle_\varphi = \langle \pi_\varphi(x)\xi_\varphi|\xi_\varphi\rangle_\varphi.$$

The representation is cyclic since $\pi_\varphi(A)\xi_\varphi = A/N_\varphi$ is norm dense in H_φ.

Appendix III

Using GNS representation theorem, we can prove Gelfand-Namimark theorem for C*-algebras. The idea is to consider

$$\pi = \oplus_\varphi \pi_\varphi : a \in A \to \oplus_\varphi \pi_\varphi(a) \in B(\oplus_\varphi H_\varphi),$$

where φ run through all states, i.e. positive linear functional of norm one, on A.

References

(1) A Course in Operator Theory, J. Conway

(2) C*-algebras by examples, K. Davidson

(3) Fundementals of the Theory of Operator Algebras I. Kadison and Ringrose

(4) Completely Bounded Maps and Operator Algebras, V. Paulsen

(5) Theory of Operator Algebras I. M. Takesaki