Introduction to Banach and Operator Algebras

Lecture 8

Zhong-Jin Ruan
University of Illinois at Urbana-Champaign

Winter School at Fields Institute
Friday January 17, 2014

Exact C*-algebras

Let $\pi : B(\ell_2) \to Q(\ell_2) = B(\ell_2)/K(\ell_2)$ be the canonical quotient map. For any C*-algebra A, we obtain a *-homomorphism

$$\pi \otimes id : B(\ell_2) \otimes^{\min} A \to Q(\ell_2) \otimes^{\min} A.$$

It is clear that $K(\ell_2) \otimes^{\min} A$ is contained in the kernel of $\pi \otimes id$.

According to Kirchberg, a C*-algebra A is an exact C*-algebra if

$$K(\ell_2) \otimes^{\min} A = \ker(\pi \otimes id),$$

i.e. if we have the short exact sequence

$$0 \to K(\ell_2) \otimes^{\min} A \hookrightarrow B(\ell_2) \otimes^{\min} A \to Q(\ell_2) \otimes^{\min} A \to 0.$$

Theorem [Kirchberg]: A C*-algebra A is exact if and only if there exists two nets of completely positive and contractive maps

$$S_\alpha : A \to M_{n(\alpha)} \text{ and } T_\alpha : M_{n(\alpha)} \to B(H)$$

such that $\|T_\alpha \circ S_\alpha(x) - x\| \to 0$ for all $x \in A$.

It follows from Kirchberg theorem that every nuclear C*-algebra is exact.

Proposition: If a C*-algebra A has the CBAP, then A is exact.

Proof: Suppose we have a net of finite rank maps $T_\alpha(x) = \sum_i f_i^\alpha(x) b_i^\alpha$ on A such that $\|T_\alpha\|_{cb} \le C < \infty$ and $T_\alpha \to id$ in the point-norm topology. Then for any $u \in \ker(\pi \otimes id) \subseteq B(\ell_2) \otimes^{\min} A$, we have

$$(id \otimes T_\alpha)(u) = \sum_i (id \otimes f_i^\alpha)(u) \otimes b_i^\alpha \to u$$

in the norm topology in $B(\ell_2) \otimes^{\min} A$. Notice that

$$\pi((id \otimes f_i^\alpha)(u)) = f_i^\alpha(\pi \otimes id)(u)) = 0.$$

This shows that each $(id \otimes T_\alpha)(u)$ is contained in $K(\ell_2) \otimes A$ and thus $u \in K(\ell_2) \otimes^{\min} A$.

Examples of Exact C^*-algebras

- For C*-algebras, we have

$$\text{Nulcearity} \quad \Rightarrow \text{CBAP} \quad \Rightarrow \text{Exactness}$$

- For any discrete group G, we have

$$\text{Amenability} \Rightarrow \text{Weakly Amenability} \Rightarrow \text{Exactness, i.e. } C^*_\lambda(G) \text{ is exact}$$

- Groups like $G = \mathbb{F}_n, \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}), G = SL(3, \mathbb{Z})$ are exact.

Non-Examples of Exact C^*-algebras

- $C^*(\mathbb{F}_n)$ for $n \geq 2$ and $B(H)$ if dim $H = \infty$.

Some Interesting Theorems

It is easy to see that if A is an exact C*-algebra, then any C*-subalgebra or subspace of A is also exact. Therefore, every C*-subalgebra of nuclear C*-algebra is exact.

Theorem [Kirchberg and Phillips 2000]: If A is a separable exact C*-algebra, then A is *-isomorphic to a C*-subalgebra of O_2.

The Cuntz algebra O_2 is the universal C*-algebra generated by isometries S_1 and S_2 such that $S_1 S_1^* + S_2 S_2^* = 1$.

It is known that the Cuntz algebra is nuclear, simple, purely infinite C*-algebra.

How about group C*-algebras ?

Roe Algebra $C_u^*(G)$

Now let G be a discrete group. Then $span\{f\lambda_s : f \in \ell_\infty(G), s \in G\}$ is a unital *-subalgebra of $B(\ell_2(G))$.

It is clearly unital. It is subalgebra since

$$(f\lambda_s)(g\lambda_t) = f\,\lambda_s g \lambda_{s^{-1}}\,\lambda_{st} = (f\,sg)\lambda_{st}.$$

It is also closed under the involution since

$$(f\lambda_s)^* = \lambda_{s^{-1}}\bar{f} = (\lambda_{s^{-1}}\bar{f}\lambda_s)\lambda_{s^{-1}} = (s^{-1}\bar{f})\,\lambda_{s^{-1}}.$$

Therefore,

$$C_u^*(G) = \overline{span\{f\lambda_s : f \in \ell_\infty(G), s \in G\}}^{\|\cdot\|} \subseteq B(\ell_2(G))$$

is a unital C*-algebra, which is called uniform Roe algebra. In fact, $C_u^*(G) = \ell_\infty(G) \rtimes G$. It contains $C_\lambda^*(G), \ell_\infty(G)$ and $K(\ell_2(G)) = c_0(G) \rtimes G$.

C*-algebra Crossed Product

Let $A \subseteq B(H)$ be a unital C*-algebra and $\alpha : G \curvearrowright A$ is an action of G on A. We can obtain a representation $\pi : A \to B(H \otimes \ell_2(G))$ given by

$$\pi(a)(\xi \otimes \delta_s) = \alpha_{s^{-1}}(a)(\xi) \otimes \delta_s$$

and an unitary represntation $\tilde{\lambda}_s : G \to B(H \otimes \ell_2(G))$

$$\tilde{\lambda}_s = 1 \otimes \lambda_s.$$

Then the reduced C*-algebra crossed product

$$A \rtimes_{\alpha,r} G = \{\sum \pi(a_s)\tilde{\lambda}_s\}^{-\|\cdot\|} \subseteq B(H \otimes \ell_2(G)).$$

To simplify notation we simply write $\sum_s \pi(a_s)\tilde{\lambda}_s$ as $\sum_s a_s \lambda_s$.

Positive Definite Schur Multipliers

A function $\phi : G \times G \to \mathbb{C}$ is a positive definite Schur multiplier if for any $s_1, \cdots, s_n \in G$, $[\phi(s_i, s_j)]$ is a positive definite matrix in $M_n(\mathbb{C})$.

Remark: If $\varphi : G \to \mathbb{C}$ is a p.d. Herz-Schur multiplier, then

$$\phi(s,t) = \varphi(s^{-1}t)$$

defines a (left invariant) Schur multiplier.

Theorem: Let $\phi : G \times G \to \mathbb{C}$. TFAE:

(1) ϕ is a p.d. Schur multiplier,

(2) the Schur map $T_\phi : [x_{s,t}] \in B(\ell_2(G)) \to [\phi(s,t)x_{s,t}] \in B(\ell_2(G))$ defines a (weak* continuous) cp map on $B(\ell_2(G))$,

(3) there exists a bounded map $\alpha : G \to \ell_2(I)$ such that

$$\phi(s,t) = \langle \alpha(s) \mid \alpha(t) \rangle = \alpha(s)^* \alpha(t).$$

General Schur Multipliers

A function $\phi : G \times G \to \mathbb{C}$ is a Schur multiplier if the Schur map

$$T_\phi : [x_{s,t}] \in B(\ell_2(G)) \to [\phi(s,t)x_{s,t}] \in B(\ell_2(G))$$

defines a (weak* continuous) cb map on $B(\ell_2(G))$. This is equivalent to say that there exists two bounded maps $\alpha, \beta : G \to \ell_2(I)$ such that

$$\phi(s,t) = \langle \alpha(t) \mid \beta(s) \rangle = \beta(s)^* \alpha(t).$$

If $\varphi : G \to \mathbb{C}$ is a completely bounded/Herz-Schur multiplier, then

$$\phi(s,t) = \varphi(s^{-1}t)$$

defines a (left invariant) Schur multiplier.

The following theorem was first observed by Guentner and Kaminker, but was finally proved by Ozawa.

Theorem [Ozawa]: Let G be a discrete group. Then TFAE:

1. G is exact, i.e the reduced group C*-algebra $C^*_\lambda(G)$ is exact;

2. for any finite subset $E \subseteq G$ and $\varepsilon > 0$, there exists a finite subset $F \subseteq G$ and a positive definite Schur multiplier $u : G \times G \to \mathbb{C}$ such that

$$|u(s,t) - 1| < \varepsilon \text{ if } s^{-1}t \in E \text{ and } u(s,t) = 0 \quad \text{if} \quad s^{-1}t \notin F.$$

3. $C^*_u(G) = \ell_\infty(G) \rtimes G$ is nuclear.

Let E be a subset of G. We define

$$\triangle_E = \{(s,t) : s^{-1}t \in E\}$$

to be a strip associated with E. In particular, if $E = \{e\}$,

$$\triangle_e = \{(s,t), s^{-1}t \in \{e\}\} = \{(s,s) : s \in G\}$$

is just the diagonal of $G \times G$. Here we are mainly interested in the finite strips, i.e. strips with finite subsets $E \subseteq G$.

Now we can restate the theorem as follows.

Theorem [Ozawa]: Let G be a discrete group. Then TFAE:

1. G is exact, i.e the reduced group C*-algebra $C^*_\lambda(G)$ is exact;

2. for any finite subset $E \subseteq G$ and $\varepsilon > 0$, there exists a finite subset $F \subseteq G$ and a positive definite Schur multiplier $\phi_{(E,\varepsilon)} : G \times G \to \mathbb{C}$ such that

$$|\phi_{(E,\varepsilon)}(s,t) - 1| < \varepsilon \text{ if } s^{-1}t \in E \text{ and } \phi_{(E,\varepsilon)}(s,t) = 0 \quad \text{if} \quad s^{-1}t \notin F,$$

(2') there exists a net of positive definite Schur multipliers $\phi_\alpha : G \times G \to \mathbb{C}$ such that
1) $\phi_\alpha \to 1$ uniformly on each finite strip $\triangle_E$
2) each ϕ_α is supported on some finite strip $\triangle_{F_\alpha}$,

3. $C^*_u(G) = \ell_\infty(G) \rtimes G$ is nuclear.

Coarse Embedding

In his study of large scale properties of finitely generated groups, Gromov introduced the notion of coarse embeddability. We recall that a metric space $(\mathcal{X}, d_{\mathcal{X}})$ is coarsely embeddable into another metric space $(\mathcal{Y}, d_{\mathcal{Y}})$ if there is a function $f : \mathcal{X} \to \mathcal{Y}$ for which there exist non-decreasing functions

$$\rho_{\pm} : \mathbb{R}_{+} \to \mathbb{R}_{+}$$

such that $\lim_{r \to +\infty} \rho_{\pm}(r) = \infty$ and

$$\rho_{-}(d_{\mathcal{X}}(x, y)) \leq d_{\mathcal{Y}}(f(x), f(y)) \leq \rho_{+}(d_{\mathcal{X}}(x, y))$$

for all $x, y \in \mathcal{X}$.

Some Equivalent Theorems

Theorem [Dadarlat and Guentner 2003]: A countable discrete group G is coarsely embeddable into a Hilbert space if and only if there exists a sequence of positive definite Schur multipliers $\phi_n : G \times G \to \mathbb{C}$ such that
1) each ϕ_n is in $C_0(G \times G, \triangle_e)$,
2) $\phi_n \to 1$ uniformly on finite strips $\triangle_E$.

We say that a Schur mulriplier ϕ is vanishing off the diagonal, $\phi \in C_0(G \times G, \triangle_e)$, if for arbitrary $\varepsilon > 0$, there exists a finite set $F \subseteq G$ such that for all $(s,t) \notin \triangle_F$, we have $|\phi(s,t)| < \varepsilon$.

Examples of Coarsely Embeddable Groups

- Amenable groups, hyperpobic groups, $SL(3, \mathbb{Z})$, exact groups

- Groups with the Haagerup property

Non-example of Coarsely Embeddable Groups

- Gromov's example of finitely generated groups with a sequence of spanders

Summaring our discussion, we have

Amenable Groups	**Exact Groups**
Groups has the HP	**Coarsely Embeddable Gr**
Consider completely bounded p.d. multipliers $\varphi : G \to \mathbb{C}$	*Consider p.d.Schur multip* $\phi : G \times G \to \mathbb{C}$.

If we have $\varphi : G \to \mathbb{C}$, then we get $\phi : G \times G \to \mathbb{C}$ with

$$\phi(s,t) = \varphi(s^{-1}t).$$

Thank you for your attention.