Crossed products from minimal dynamical systems on the connected odd dimensional spaces

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(L–Phillips 2004—2010) Let Ω be a finite dimensional and infinite compact metric space let β be a minimal homeomorphism and let $A = C(\Omega) \rtimes_\beta \mathbb{Z}$. Then A has tracial rank zero if and only $\rho_A(K_0(A))$ is dense in $\mathrm{Aff}(T(A))$.

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(Toms-Winter—2009) Let Ω be a finite dimensional compact metric space, let β be a minimal homeomorphism and let $A = C(\Omega) \rtimes_\beta \mathbb{Z}$. Then A is $\mathcal{Z}$-stable.

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Corollary

Let Ω be a finite dimensional compact metric space, let β be a homeomorphism. If (Ω, β) is unique ergodic, then $C(\Omega) \rtimes_\beta \mathbb{Z}$ is in $\mathcal{A}$.

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What happens that $C(\Omega) \rtimes_\beta \mathbb{Z}$ does not have many projections (Ω is connected) and have many tracial states? For example, what happens when $\Omega = S^{2n+1}$ and (S^{2n+1}, β) has many β-invariant measures?

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(Comm. Math. Phys, **275** *(2005), 425–471)*

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The C^*-algebra A_x is locally AH. Moreover, A_x is isomorphic to a unital simple AH-algebra with slow dimension growth. (L–2014)

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$$K_1(A_1) \cong K_1(A_2), \ (\beta_1)_* = (\beta_2)_* \text{ on } K_0(C(RP^{2n+1})) \\ \text{and } T(A_1) = T(A_2).$$

The Research Center for Operator Algebras at East China Normal University is now recruiting postdocs.