Expansive Automorphisms of Totally Disconnected, Locally Compact Groups

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Throughout the talk:

G a totally disconnected, locally compact group

Defn Automorphism $\alpha\colon G \to G$ is **expansive** if

$$\bigcap_{k\in\mathbb{Z}} \alpha^k(V) = \{1\}$$

for some identity neighborhood $V \subseteq G$.

[Without loss of generality V a compact open subgroup]

Structure of talk:

I. General theory of expansive automorphisms

(**II.** Special case of p-adic Lie groups)

§1 Expansive automorphisms: basic facts

Defn Automorphism $\alpha\colon G \to G$ is **expansive** if

$$V_0 \;:=\; \bigcap_{k\in\mathbb{Z}} \alpha^k(V) \;=\; \{1\}$$

for some compact open subgroup $V \subseteq G$.

Ex If an automorphism $\alpha\colon G \to G$ is **contractive** (i.e., $\alpha^n(x) \to 1$ as $n \to \infty$ for all $x \in G$), then α is expansive.

In fact, G has a compact open subgroup V such that

$$V \supseteq \alpha(V) \supseteq \alpha^2(V) \supseteq \cdots$$

and $\bigcap_{k=0}^{\infty} \alpha^k(V) = \{1\}$ (Siebert 1986).

Ex $\alpha(x,y) := (px, p^{-1}y)$ is an expansive automorphism of $\mathbb{Q}_p \times \mathbb{Q}_p$, as

$$\bigcap_{k\in\mathbb{Z}} \alpha^k(\mathbb{Z}_p\times\mathbb{Z}_p) \;=\; \bigcap_{k\in\mathbb{Z}} (p^k\mathbb{Z}_p\times p^{-k}\mathbb{Z}_p) \;=\; \{(0,0)\}.$$

More generally: If $\alpha\colon G \to G$ and $\beta\colon H \to H$ are contractive, then $\alpha \times \beta^{-1}$ is an expansive automorphism of $G \times H$.

The **contraction group** of $\alpha \in \mathrm{Aut}(G)$ is

$$U_\alpha := \{x \in G \colon \alpha^n(x) \to 1 \text{ as } n \to \infty\}$$

U_α is a subgroup of G; need not be closed

Basic Lemma (Link between contractive and expansive automorphisms)
If $\alpha \in \mathrm{Aut}(G)$ is expansive, then

$$U_\alpha U_{\alpha^{-1}}$$

is an open subset of G.

Rem (a) $U_\alpha U_{\alpha^{-1}}$ need not be a subgroup
(b) U_α need not normalize $U_{\alpha^{-1}}$
(c) It can happen that $U_\alpha \cap U_{\alpha^{-1}} \neq \{1\}$.

Main consequence *If $\alpha \in \mathrm{Aut}(G)$ is expansive, $H \subseteq G$ a subgroup which is not open in G, then $H \cap U_\alpha \subsetneq U_\alpha$ or $H \cap U_{\alpha^{-1}} \subsetneq U_{\alpha^{-1}}$*

Otherwise $H \supseteq (H \cap U_\alpha)(H \cap U_{\alpha^{-1}}) = U_\alpha U_{\alpha^{-1}}$, i.e. H is an identity neighborhood, thus open

Basic Lemma

$\alpha \in \mathrm{Aut}(G)$ expansive $\Rightarrow U_\alpha U_{\alpha^{-1}}$ open in G

If $V \subseteq G$ is a compact open subgroup, write

$$V_- := \bigcap_{k=0}^{\infty} \alpha^{-k}(V), \quad V_{--} := \bigcup_{k=0}^{\infty} \alpha^{-k}(V_-).$$

Lemma *If $\alpha \in \mathrm{Aut}(G)$ is expansive and $V \subseteq G$ a compact open subgroup such that $V_0 := \bigcap_{k \in \mathbb{Z}} \alpha^k(V) = \{1\}$, then $U_\alpha = V_{--}$.*

In fact, $V_{--} = U_\alpha V_0$ for each c.o. subgroup V by Baumgartner-Willis (2004), Prop. 3.16.

Proof of Basic Lemma There exists a c.o. subgroup $V \subseteq G$ such that $V_0 = \{1\}$. After replacing V with $\bigcap_{k=0}^{n} \alpha^k(V)$ for some n, may assume $V = V_+ V_-$. Then

$$U_\alpha U_{\alpha^{-1}} = V_{--} V_{++} \supseteq V_- V_+ = V.$$

Hence $U_\alpha U_{\alpha^{-1}}$ is an identity neighborhood and hence open.

We also deduce a lemma by Siebert (1989):

Lemma *If* $\alpha \in \mathrm{Aut}(G)$ *is expansive, then* U_α *can be made locally compact, i.e., its topology can be refined to a locally compact group topology* τ^* *such that* α *remains contractive on* $U_\alpha^* := (U_\alpha, \tau^*)$*.*

Proof (sketch) For V as above, give $U_\alpha = V_{--} = \bigcup_{k=0}^{\infty} \alpha^{-k}(V_-)$ the group topology τ^* making V_- a compact open subgroup.

Further simple facts

(a) If $\alpha \in \mathrm{Aut}(G)$ is expansive, then also $\alpha|_H$ for each α-stable closed subgroup $H \subseteq G$.

(b) If $\alpha \in \mathrm{Aut}(G)$ is expansive, then G is metrizable (cf. Lam (1970)).

Take a c.o. subgroup $V \subseteq G$ with $\bigcap_{k \in \mathbb{Z}} \alpha^k(V) = \{1\}$. Then $\bigcap_{k=-n}^{n} \alpha^k(V)$, $n \in \mathbb{N}$, is a countable basis of identity neighborhoods.

§2 Main Results

Theorem A (G.-Raja 2013) *Let* $\alpha \in \mathrm{Aut}(G)$ *and* $N \subseteq G$ *be an* α*-stable closed normal subgroup. Then* α *is expansive if and only if both* $\alpha|_N$ *and the induced automorphism*

$$\overline{\alpha}\colon G/N \to G/N, \quad gN \mapsto \alpha(g)N$$

are expansive.

Main point: α *expansive* $\Rightarrow$ $\overline{\alpha}$ *expansive*

The second main result concerns the divisible part D_α of the contraction group U_α, for $\alpha\colon G \to G$ an expansive automorphism.

Recall from G.-Willis (2010):

If $\alpha \in \mathrm{Aut}(G)$ is contractive, then the set

$$\mathrm{tor}(G)$$

of torsion elements is a characteristic subgroup (a torsion group of finite exponent); the set

$$\mathrm{div}(G)$$

of all divisible elements is a subgroup; and

$$G = \mathrm{div}(G) \times \mathrm{tor}(G)$$

internally as a topological group. Moreover,

$$\mathrm{div}(G) = G_{p_1} \times \cdots \times G_{p_n}$$

with certain α-stable p-adic Lie groups G_p.

Now α expansive $\Rightarrow U_\alpha^*$ locally compact, so

$U_\alpha^* = D_\alpha \times T_\alpha$ with $D_\alpha := \mathrm{div}\, U_\alpha^*$ and $T_\alpha := \mathrm{tor}\, U_\alpha^*$.
Although U_α need not be closed, we have:

Theorem B (G.-Raja (2013)) *If U_α can be made locally compact (e.g., if $\alpha \in \mathrm{Aut}(G)$ is expansive), then D_α is closed in G.*

§3 Tools for the proof of Theorem A

Observation (G.-Willis (2010)) *If α is a contractive automorphism of $G \neq \{1\}$ and*

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = \{1\}$$

are α-stable closed subgroups of G such that G_j is normal in G_{j-1} for $j \in \{1, \ldots, n\}$, then the module $\Delta(\alpha^{-1})$ is an integer ≥ 2 and n is bounded by the number of prime factors of $\Delta(\alpha^{-1})$. Conclude:

Lemma *Let $\alpha \in \mathrm{Aut}(G)$ be expansive,*

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

be α-stable closed subgroups of G such that G_j is normal in G_{j-1} for $j \in \{1, \ldots, n\}$ and

$$J := \Big\{ j \in \{1, \ldots, n\} \colon G_j \text{ is not open in } G_{j-1} \Big\}.$$

Then $\#J$ is bounded by the number of prime factors of $\Delta(\alpha^{-1}|_{U_\alpha^*})\Delta(\alpha|_{U_{\alpha^{-1}}^*})$.

In fact, if G_j is not open in G_{j-1}, then $G_j \cap U_\alpha \subsetneq G_{j-1} \cap U_\alpha$ or $G_j \cap U_{\alpha^{-1}} \subsetneq G_{j-1} \cap U_{\alpha^{-1}}$

For compact (pro-finite) G, see Willis (2012) for the following result:

Prop. *If G is pro-discrete and $\alpha \in \mathrm{Aut}(G)$, then $G = \varprojlim G/N$ for N in a filter basis of α-stable closed normal subgroups of G s.t. the automorphism induced on G/N is expansive.*

Proof. For each open normal subgroup $V \subseteq G$,

$$V_0 := \bigcap_{k \in \mathbb{Z}} \alpha^k(U)$$

is an α-stable closed normal subgroup of G such that α induces an expansive automorphism on G/V_0. Moreover, $G = \varprojlim G/V_0$.

By Baumgartner/Willis (2004), the Levi factor

$$M_\alpha := \Big\{x \in G \colon \{\alpha^k(x) \colon k \in \mathbb{Z}\} \text{ is relatively compact}\Big\}$$

is a closed subgroup of G which normalizes U_α.

$$M_\alpha := \Big\{x \in G\colon \{\alpha^k(x)\colon k \in \mathbb{Z}\} \text{ is relatively compact}\Big\}$$

Lemma (G.-Raja) $\alpha \in \mathrm{Aut}(G)$ *is expansive if and only if* $\alpha|_{M_\alpha}$ *is expansive.*

Proof. Let $V \subseteq M_\alpha$ be open with

$$\bigcap_{k\in\mathbb{Z}} \alpha^k(V) = \{1\}.$$

Choose a c.o. subgroup $W \subseteq G$ such that

$$W \cap M_\alpha \subseteq V.$$

If $x \in \bigcap_{k\in\mathbb{Z}} \alpha^k(W) =: I$, then $\alpha^k(x) \in W$ for each k and thus $x \in M_\alpha$ (since W is compact). Thus $I \subseteq W \cap M_\alpha \subseteq V$ and thus

$$I = \bigcap_{k\in\mathbb{Z}} \alpha^k(I) \subseteq \bigcap_{k\in\mathbb{Z}} \alpha^k(V) = \{1\}.$$

§4 Proof of Theorem A

Let $\alpha \in \mathrm{Aut}(G)$ be expansive. To show: $\overline{\alpha}$ on G/N is expansive.

Use $q\colon G \to G/N$, $q(x) := xN$.

Without loss $G/N = M_{\overline{\alpha}}$.

Indeed, only need $\overline{\alpha}$ is expansive on $M_{\overline{\alpha}}$. So replace G with $q^{-1}(M_{\overline{\alpha}})$.

Without loss G/N is compact.

Let $V \subseteq G/N$ be a c.o. subgroup tidy for $\overline{\alpha}$. As all two-sided $\overline{\alpha}$-orbits are relatively compact, $V = V_+ = V_-$ and thus $\overline{\alpha}(V) = V$. Now replace G with $q^{-1}(V)$.

Since G/N is profinite and metrizable, there are $\overline{\alpha}$-stable closed normal subgroups

$$H_1 \supseteq H_2 \supseteq \cdots$$

of G/N such that the automorphism α_n induced by $\overline{\alpha}$ on $(G/N)/H_n$ is expansive and

$$G/N = \varprojlim (G/N)/H_n.$$

Then $\bigcap_{n=1}^{\infty} H_n = \{1\}$, so $\bigcap_{n=1}^{\infty} q^{-1}(H_n) = N$.

§4 Proof of Theorem A

Let $\alpha \in \mathrm{Aut}(G)$ be expansive. Show: $\overline{\alpha}$ on G/N is expansive if G/N is compact. Use $q\colon G \to G/N$.

Since G/N is profinite and metrizable, there are $\overline{\alpha}$-stable closed normal subgroups

$$H_1 \supseteq H_2 \supseteq \cdots$$

of G/N such that the automorphism α_n induced by $\overline{\alpha}$ on $(G/N)/H_n$ is expansive and

$$G/N = \varprojlim (G/N)/H_n.$$

Then $\bigcap_{n=1}^{\infty} H_n = \{1\}$, so $\bigcap_{n=1}^{\infty} q^{-1}(H_n) = N$.

Since $q^{-1}(H_1) \triangleright q^{-1}(H_2) \triangleright \cdots$, there is n such that $q^{-1}(H_m)$ is open in $q^{-1}(H_n)$ for all $m \geq n$.

Thus $q^{-1}(H_n) \cap U_\alpha, q^{-1}(H_n) \cap U_{\alpha^{-1}} \subseteq q^{-1}(H_m)$ for $m \geq n$ and hence

$$q^{-1}(H_n) \cap U_\alpha, q^{-1}(H_n) \cap U_{\alpha^{-1}} \subseteq \bigcap_{m \geq n} q^{-1}(H_m) = N.$$

Thus $(q^{-1}(H_n) \cap U_\alpha)(q^{-1}(H_n) \cap U_{\alpha^{-1}}) \subseteq N$, whence N is open in $q^{-1}(H_n)$ and $H_n = q^{-1}(H_n)/N$ is discrete, hence finite.

Since $\bigcap_{m \geq n} H_m = \{1\}$, find $m \geq n$ such that $H_m = \{1\}$. Since $\overline{\alpha}$ corresponds to α_m on $(G/N)/H_m \cong G/N$, it is expansive.

§5 Tools for the proof of Theorem B

Thm B. *If U_α can be made locally compact (e.g., if $\alpha \in$ Aut(G) is expansive), then $D_\alpha :=$ div(U_α^*) is closed in G.*

We shall use the **nub** U_0 of $\alpha \in$ Aut(G), defined as the intersection of all compact open subgroups tidy for α.

Facts (a) *The closure of U_α is $\overline{U_\alpha} = U_\alpha U_0$.*
(b) *$U_0 \cap U_\alpha$ is dense in U_0.*

See Baumgartner-Willis (2004) for (a), Willis (2012) for (b).

Consequence (G.-Raja) *If U_α can be made locally compact (e.g., $\alpha \in$ Aut(G) expansive), then $U_0 \cap U_\alpha = U_0 \cap T_\alpha$ and thus $U_0 = \overline{U_0 \cap T_\alpha}$.*

$U_0 \cap U_\alpha^*$ lcp, hence $U_0 \cap U_\alpha$ can be made lcp, hence $U_0 \cap U_\alpha = DT$ with D divisible, T torsion. Then $D = \{1\}$ as U_0 (like any finite or profinite group) has no divisible elements. So $U_0 \cap U_\alpha = T = U_0 \cap T_\alpha$.

§6 Proof of Theorem B

Thm B. *If U_α can be made locally compact* (*e.g., if $\alpha \in$* Aut(G) *is expansive*), *then $D_\alpha :=$ div(U_α^*) is closed in G.*

Proof. Replacing G with $\overline{U_\alpha}$, w.l.o.g. $G = \overline{U_\alpha} = U_\alpha U_0$. As U_0 normalizes U_α, have $U_\alpha \triangleleft G$.

Since D_α and T_α are characteristic in U_α, also $D_\alpha \triangleleft G$ and $T_\alpha \triangleleft G$. Hence $\overline{T_\alpha} \triangleleft G$.

Since $\overline{T_\alpha}$ is a torsion group (as T_α has finite exponent) and D_α is torsion-free, we have

$$D_\alpha \cap \overline{T_\alpha} = \{1\}.$$

Moreover, $G = \overline{U_\alpha} = U_\alpha U_0 = D_\alpha T_\alpha \overline{U_0 \cap T_\alpha} = D_\alpha \overline{T_\alpha}$. Hence $G = D_\alpha \times \overline{T_\alpha}$ as an abstract group. Thus

$$D_\alpha^* \times \overline{T_\alpha} \to G, \quad G, \quad (x, y) \mapsto xy$$

is continuous and an isomorphism of abstract groups, hence an isomorphism of topological groups (by the Open Mapping Theorem), as the groups on both sides are locally compact and σ-compact. Notably, D_α is closed in G.

The proof showed more:

Theorem C. (G.-Raja (2013)) *If* U_α *can be made locally compact* (*e.g., if* $\alpha \in \mathrm{Aut}(G)$ *is expansive*), *then*

$$\overline{U_\alpha} = D_\alpha \times \overline{T_\alpha}$$

(*internally*) *as a topological group.*

§7 Expansive automorphisms of Lie groups

Let G be a Lie group over a totally disconnected local field $\mathbb{K}$ (e.g., $\mathbb{Q}_p$) and $\alpha\colon G \to G$ be a $\mathbb{K}$-analytic automorphism. Then $\beta := T_1(\alpha)$ is a linear automorphism of the Lie algebra $\mathfrak{g} := T_1(G)$.

For $\rho > 0$, define

$$\mathfrak{g}_\rho := \mathfrak{g} \cap \bigoplus_{|\lambda|=\rho} (\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}})_\lambda,$$

where $\overline{\mathbb{K}}$ is an algebraic closure, $|.|$ the unique extension of the absolute value on $\mathbb{K}$ to $\overline{\mathbb{K}}$ and

$$(\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}})_\lambda \quad \text{for } \lambda \in \overline{\mathbb{K}}$$

the generalized eigenspace of $\beta \otimes_{\mathbb{K}} \mathrm{id}_{\overline{\mathbb{K}}}$. Then

$$\mathfrak{g} = U_\beta \oplus M_\beta \oplus U_{\beta^{-1}}$$

with $M_\beta = \mathfrak{g}_1$,

$$U_\beta = \bigoplus_{\rho<1} \mathfrak{g}_\rho \text{ and } U_{\beta^{-1}} = \bigoplus_{\rho>1} \mathfrak{g}_\rho.$$

Theorem D (G.-Raja (2013))

(a) *If* α *is expansive, then* $M_\beta = \{0\}$, *i.e.,* $|\lambda| \neq 1$ *for all eigenvalues* $\lambda \in \overline{\mathbb{K}}$ *of* $\beta \otimes_{\mathbb{K}} \mathrm{id}_{\overline{\mathbb{K}}}$.

(b) *If* U_α *closed, then* α *expansive iff* $M_\beta = \{0\}$.

Proof. (b) If $M_\beta \neq \{0\}$ and $U \subseteq G$ is an identity neighborhood, then U contains a so-called center manifold $W \subseteq G$, which can be chosen as an α-stable Lie subgroup with Lie algebra M_β, by the theory of time-discrete $\mathbb{K}$-analytic dynamical systems (G. (2013)). Then $W \subseteq \bigcap_{k \in \mathbb{Z}} \alpha^k(U)$ and thus α is not expansive.

(a) If U_α is closed then M_α is a Lie subgroup with Lie algebra M_β (cf. G. (2008)), which is $\{0\}$ iff M_α is discrete. Now apply next lemma.

Lemma (G.-Raja (2013)) *If* G *is a t.d.l.c. group,* $\alpha \in \mathrm{Aut}(G)$ *and* U_α *is closed, then* α *is expansive iff* M_α *is discrete.*

Rem. If α is expansive, then $\mathfrak{g}$ is nilpotent (this follows with an exercise from Bourbaki). Hence, if $\mathbb{K} = \mathbb{Q}_p$, then G has an open nilpotent subgroup. Can it be chosen α-stable?

Lemma (G.-Raja (2013)) *If G is a t.d.l.c. group, $\alpha \in \mathrm{Aut}(G)$ and U_α is closed, then α is expansive iff M_α is discrete.*

Proof. $\Rightarrow$ Let V be a c.o. subgroup of G such that $V_0 = \{1\}$. Since U_α is closed, M_α has a c.o. subgroup $W \subseteq M_\alpha \cap V$ which is tidy for α and hence α-stable. Thus $W \subseteq V_0$ and thus $W = \{1\}$, whence M_α is discrete.

$\Leftarrow$ Since U_α is closed, the set $U_\alpha M_\alpha U_{\alpha^{-1}}$ is an open identity neighborhood in G and the product map

$$U_\alpha \times M_\alpha \times U_{\alpha^{-1}} \to U_\alpha M_\alpha U_{\alpha^{-1}},\ (x,y,z) \mapsto xyz$$

is a homeomorphism (G. (2005); cf. Wang (1984) for the p-adic case). Hence, if M_α is discrete, then $U_\alpha U_{\alpha^{-1}}$ is open in G and

$$\bigcap_{k \in \mathbb{Z}} \alpha^k(VW) = \{1\}$$

for all compact open identity neighborhoods $V \subseteq U_\alpha$, $W \subseteq U_{\alpha^{-1}}$.

Rem. If α is expansive, then $\mathfrak{g}$ is nilpotent. Hence, if $\mathbb{K} = \mathbb{Q}_p$, then G has an open nilpotent subgroup. Can it be chosen α-stable?

Theorem E (G.-Raja (2013)) *Let α be an expansive automorphism of a p-adic Lie group G. If G is linear in the sense that there exists an injective continuous homomorphism $G \to \mathrm{GL}_n(\mathbb{Q}_p)$, then G has an α-stable, open nilpotent subgroup.*

Rem For α expansive and G a **closed** subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$, can show $U_\alpha U_{\alpha^{-1}}$ is a **subgroup** of G (which is α-stable and nilpotent).

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