Spectral Synthesis and Ideal Theory
Lecture 2

Eberhard Kaniuth

University of Paderborn, Germany

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Synthesis Notions

A a regular and semisimple commutative Banach algebra. For a closed subset E of $\Delta(A)$, let

$$j(E) = \{a \in A : \widehat{a} \text{ has compact support disjoint from E}\}.$$

Then, if I is any ideal in A with $h(I) = E$,

$$j(E) \subseteq I \subseteq k(E).$$

Definition

E is called a *set of synthesis* or *spectral set* if $\overline{j(E)} = k(E)$ (equivalently, $I = k(E)$ for any closed ideal I with $h(I) = E$).

We say that *spectral synthesis holds* for A if every closed subset of $\Delta(A)$ is a set of synthesis.

Definition

$E \subseteq \Delta(A)$ closed is called *Ditkin set* if $a \in \overline{aj(E)}$ for every $a \in k(E)$. Thus

- Every Ditkin set is a set of synthesis

- $\emptyset$ is a Ditkin set if and only if given $a \in A$ and $\epsilon > 0$, there exists $b \in A$ such that $\widehat{b}$ has compact support and $\|a - ab\| \leq \epsilon$ (in this case we also say that A *satisfies Ditkin's condition at infinity*)

A is called *Tauberian* if the set of all $a \in A$ such that $\widehat{a}$ has compact support, is dense in A. Thus

- A is Tauberian if and only if $\emptyset$ is a set of synthesis.

When does Spectral Synthesis hold for A?

Spectral synthesis holds for $C_0(X)$, X a locally compact Hausdorff space

Spectral synthesis does not hold for $C^n[a,b]$, $n \geq 1$: singletons $\{t\}$, $t \in [a,b]$, are not sets of synthesis

Remark

Suppose that spectral synthesis holds for A. Then $a \in \overline{aA}$ for each $a \in A$.
Proof:

Let $E = \{\varphi \in \Delta(A) : \varphi(a) = 0\}$. Then E is closed in $\Delta(A)$ and $E = h(\overline{aA})$. Thus $a \in k(E) = \overline{aA}$ since E is of synthesis.

The condition that $a \in \overline{aA}$ for every $a \in A$ is satisfied, if A has an approximate identity.

Lemma

Let A be a regular and semisimple commutative Banach algebra and E an open and closed subset of $\Delta(A)$.

1. *If A is Tauberian and $a \in \overline{aA}$ for every $a \in k(E)$, then E is a set of synthesis.*
2. *If A satisfies Ditkin's condition at infinity, then E is a Ditkin set.*

Proof of (2) Have to show that $a \in \overline{aj(E)}$ for each $a \in k(E)$:

• E open and closed $\Longrightarrow$

$$h(j(E) + j(\Delta(A) \setminus E)) = E \cap (\Delta(A) \setminus E) = \emptyset$$

and hence $j(\emptyset) \subseteq j(E) + j(\Delta(A) \setminus E)$

• $\emptyset$ Ditkin $\Rightarrow$ for every $a \in A$, there exist sequences $(u_n)_n \subseteq j(E)$ and $(v_n)_n \subseteq j(\Delta(A) \setminus E)$ such that $a(u_n + v_n) \to a$

• let $a \in k(E)$: then $\widehat{av_n} = \widehat{a}\widehat{v_n}$ vanishes on E and on $\Delta(A) \setminus E$, hence $av_n = 0$. So $a = \lim_{n\to\infty} au_n \in \overline{aj(E)}$, as required.

From the first assertion of the lemma and the above remark it follows

Corollary

Suppose that $\Delta(A)$ is discrete and A is Tauberian. Then spectral synthesis holds for A if and only if $a \in \overline{aA}$ for each $a \in A$.

Corollary

Let G be a compact abelian group. Then spectral synthesis holds for $L^1(G)$.

Proof.

- $L^1(G)$ has an approximate identity
- $L^1(G)$ is Tauberian
- $\widehat{G} = \Delta(L^1(G))$ is discrete since G is compact. □

The Example of L. Schwartz

Theorem

For $n \geq 3$, the sphere $S^{n-1} = \{y \in \mathbb{R}^n : \|y\| = 1\} \subseteq \Delta(L^1(\mathbb{R}^n))$ fails to be a set of synthesis for $L^1(\mathbb{R}^n)$.

Remark

(1) L. Schwartz [Sur une propriété de synthèse spectrale dans les groupes noncompacts, C.R. Acad. Sci. Paris **227** (1948), 424-426] proved this result for $n = 3$, but the proof works for all $n \geq 3$.

(2) $S^1 \subseteq \mathbb{R}^2$ is a set of synthesis for $L^1(\mathbb{R}^2)$ [C. Herz, *Spectral synthesis for the circle*, Ann. Math. **68** (1958), 709-712]

Proof of Schwartz' Theorem

Identify $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$ through $y \to \gamma_y$, where $\gamma_y(x) = \langle x, y \rangle$ for $x \in \mathbb{R}^n$.

- $\widehat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x,y \rangle} dx, \quad f \in L^1(\mathbb{R}^n)$
- $\check{g}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(y) e^{i\langle x,y \rangle} dy, \quad g \in L^1(\widehat{\mathbb{R}^n})$
- $f \in L^1(\widehat{\mathbb{R}^n}) \cap L^2(\widehat{\mathbb{R}^n})$ and $\check{f} \in L^1(\mathbb{R}^n)$, then $(\check{f})^\wedge = f$ in $L^2(\mathbb{R}^n)$, hence $(\check{f})^\wedge(x) = f(x)$ for all $x \in \mathbb{R}^n$ if f is continuous

Lemma

Let $D(\mathbb{R}^3)$ denote the set of all functions in $L^1(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$ with the property that all partial derivatives exist and are in $L^1(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$. Then $\widehat{f} \in L^1(\mathbb{R}^3)$ and $(\check{f})^\wedge = f$ for every $f \in D(\mathbb{R}^3)$.

Lemma

Let $S = S^2$ *and* $I = k(S) \subseteq L^1(\mathbb{R}^n)$, *and*

$$J = \left\{ f \in I : \widehat{f} \in D(\mathbb{R}^n) \text{ and } \frac{\partial \widehat{f}}{\partial y_1} = 0 \text{ on } S \right\}.$$

Then $\overline{J}$ *is an ideal in* $L^1(\mathbb{R}^3)$ *and* $h(J) = S$.

To show that $\overline{J} \neq I$, it suffices to construct a bounded linear functional F on $L^1(\mathbb{R}^3)$ such that $F(J) = \{0\}$, but $F(I) \neq \{0\}$. Such an F can be constructed as follows:

There exists a unique probability measure μ on S, which is invariant under orthogonal transformations.

Define a function ϕ on $\mathbb{R}^3$ by

$$\phi(x) = \int_S e^{-i\langle x,y\rangle} d\mu(y).$$

Then the function $x \to x_1\phi(x)$ on $\mathbb{R}^3$ is continuous and bounded. More precisely, it can be shown that

$$|x_1\phi(x)| \leq \|x\| \cdot |\phi(x)| \leq \frac{4\pi}{3}, \quad x \in \mathbb{R}^3.$$

The required functional F can now be defined by

$$F(f) = \int_{\mathbb{R}^3} f(x) x_1 \phi(x)\, dx, \quad f \in L^1(\mathbb{R}^3).$$

Since

$$\frac{\partial \widehat{f}}{\partial y_1}(y) = (-ix_1 f(x))^\wedge(y) = \int_{\mathbb{R}^3} (-ix_1) f(x) e^{-i\langle x,y\rangle} dx,$$

we have

$$i \int_S \frac{\partial \widehat{f}}{\partial y_1}(y)\, d\mu(y) = \int_S \left(\int_{\mathbb{R}^3} x_1 f(x) e^{-i\langle x,y\rangle} dx \right) d\mu(y)$$

$$= \int_{\mathbb{R}^3} x_1 f(x) \left(\int_S e^{-i\langle x,y\rangle} d\mu(y) \right) dx = \int_{\mathbb{R}^3} f(x) x_1 \phi(x) dx = F(f).$$

Thus $F(f) = 0$ for every $f \in J$.

To show that $F(I) \neq \{0\}$, consider the function

$$f(x) = (\sqrt{2})^3 e^{-\|x\|^2} - e^{1/4} e^{-\|x\|^2/2}, \quad x \in \mathbb{R}^3.$$

Then $f \in L^1(\mathbb{R}^3)$, and

$$\widehat{f}(y) = e^{-\|y\|^2/4} - e^{1/4} e^{-\|y\|^2/2}.$$

Hence $\widehat{f}(y) = 0$ if $\|y\| = 1$, i.e. $f \in I$.

We claim that $F(L_a f) \neq 0$ for some $a \in \mathbb{R}^3$ (note that $L_a f \in I$ since I is a closed ideal). For arbitrary f, we have

$$\widehat{L_a f}(y) = e^{i\langle a,y\rangle} \widehat{f}(y) \Longrightarrow \frac{\partial \widehat{L_a f}}{\partial y_1}(y) = e^{i\langle a,y\rangle} \left[i\, a_1 \widehat{f}(y) + \frac{\partial \widehat{f}}{\partial y_1}(y) \right].$$

If $f \in I$, then $\widehat{f}(y) = 0$ for $y \in S$, and hence

$$F(L_a f) = i \int_S \frac{\partial \widehat{L_a f}}{\partial y_1}(y) d\mu(y) = i \int_S e^{i\langle a,y\rangle} \frac{\partial \widehat{f}}{\partial y_1}(y)\, d\mu(y).$$

Now, for our special function f,

$$\frac{\partial \widehat{f}}{\partial y_1}(y) = -\frac{1}{2}\, y_1 e^{-\|y\|^2/4} + y_1 e^{1/4} e^{-\|y\|^2/2}$$

and hence, for $y \in S$,

$$\frac{\partial \widehat{f}}{\partial y_1}(y) = \frac{1}{2}\, y_1 e^{-1/4} y_1.$$

Finally, take $a = (\pi, 0, 0)$; then with $c = \frac{1}{2} e^{-1/4}$,

$$\begin{aligned} F(L_a f) &= i\, c \int_S e^{i\pi y_1} y_1 d\mu(y) \\ &= i\, c \int_S y_1 \cos(\pi y_1)\mu(y) - c \int_S y_1 \sin(\pi y_1)\mu(y). \end{aligned}$$

The first integral is zero since $(y_1, y_2, y_3) \to (-y_1, y_2, y_3)$ is an orthogonal transformation. So

$$F(L_a f) = c \int_S y_1 \sin(\pi y_1)\mu(y).$$

Since $y_1 \sin(\pi y_1) > 0$ whenever $y_1 \neq 0, 1, -1$, it follows that $F(L_a f) \neq 0$.

Theorem

Let $I = k(S^{n-1}) \subseteq L^1(\mathbb{R}^n)$*, and for* $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$*, let* I^k *denote the closed ideal of* $L^1(\mathbb{R}^n)$ *generated by all convolution products* $f_1 * f_2 * \ldots * f_k$*,* $f_j \in I$*. Then*

$$I = I^1 \supseteq I^2 \supseteq \ldots \supseteq I^{\lfloor \frac{n+1}{2} \rfloor} = \overline{j(S^{n-1})}.$$

- *All the inclusions are proper*
- *The ideals* I^k *are the only rotation invariant closed ideals of* $L^1(\mathbb{R}^n)$ *with hull equal to* S^{n-1}*.*

N.Th. Varopoulos, *Spectral synthesis on spheres*, Math. Proc. Camb. Phil. Soc. **62** (1966), 379-387.

Injection Theorem for Spectral Sets

A a regular and semisimple commutative Banach algebra, I a closed ideal of A and $i : \Delta(A/I) \to \Delta(A)$ the usual embedding.

Theorem

Let E be a closed subset of $\Delta(A/I)$.

- *If $i(E)$ is a set of synthesis (Ditkin set) for A, then E is a set of synthesis for A/I.*
- *Suppose that E is a set of synthesis for A/I and $h(I)$ is a set of synthesis for A. Then $i(E)$ is a set of synthesis for A.*

Remark

In the second statement of the theorem, the hypothesis on $h(I)$ cannot be dropped, and the analogue for Ditkin sets requires some additional strong hypothesis on A.

Unions of sets of synthesis and Ditkin sets

Let A be a regular and semisimple commutative Banach algebra.

Theorem

Let E und F be closed subsets of $\Delta(A)$ such that $E \cap F$ is a Ditkin set. Then $E \cup F$ is a set of synthesis if and only if both E and F are sets of synthesis.

Theorem

Let $E_1, E_2, \ldots \subseteq \Delta(A)$ be Ditkin sets. If $\bigcup_{i=1}^{\infty} E_i$ is closed in $\Delta(A)$, then $\bigcup_{i=1}^{\infty} E_i$ is a Ditkin set.

Problems

Union Problem: Let $E, F \subseteq \Delta(A)$ be sets of synthesis. Is then $E \cup F$ also a set of synthesis?

The C-set/S-set Problem: Is every set of synthesis a Ditkin set? (Ditkin sets are sometimes called C-sets, C referring to Calderon)

Since finite unions of Ditkin sets are Ditkin sets, an affirmative answer to the C-set/S-set problem implies an affirmative answer to the union problem.

In general, the answer to both questions is negative!

Both problems are open for $L^1(G)$, G a noncompact locally compact abelian group, even for $G = \mathbb{Z}$.

The Mirkil Algebra

Definition

Identify $[-\pi, \pi[$ with the circle $\mathbb{T}$, and let M be the space of all $f \in L^2(\mathbb{T})$ such that f is continuous on the interval $[-\pi/2, \pi/2]$. Endow M with the norm

$$\|f\| = \|f\|_2 + \|f|_{[-\pi/2,\pi/2]}\|_\infty$$

and convolution.

M is a regular and semisimple commutative Banach algebra, and the spectrum $\Delta(M)$ can be identified with $\mathbb{Z}$ via $n \to \varphi_n$, where

$$\varphi_n(f) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\, e^{-int} dt, \quad f \in M.$$

The algebra M shows that in the general Banach algebra context the answer to both problems is negative:

• $4\mathbb{Z}$ and $4\mathbb{Z}+2$ are both sets of synthesis, but their union $2\mathbb{Z}$ is not of synthesis

• $4\mathbb{Z}$ and $4\mathbb{Z}+2$ fail to be Ditkin sets

• Every finite subset of $\Delta(M)$ is a set of synthesis, but not a Ditkin set (in particular, $\emptyset$ is not Ditkin).

H. Mirkil, *A counterexample to discrete spectral synthesis*, Compos. Math. **14** (1960), 269-273.

A. Atzmon, *Spectral synthesis in regular Banach algebras*, Israel J. Math. **8** (1970), 197-212.

C.R. Warner, *Spectral synthesis in the Mirkil algebra*, J. Math. Anal. Appl. **167** (1992), 176-182.

Examples

(1) Every closed convex set in $\mathbb{R}^n$ is set of synthesis for $L^1(\mathbb{R}^n)$

(2) Let $D = \{y \in \mathbb{R}^n : \|y\| < 1\}$: then $\mathbb{R}^n \setminus D$ is a set of synthesis for $L^1(\mathbb{R}^n)$.

(3) $\overline{D} = \{y \in \mathbb{R}^n : \|y\| \leq 1\}$ is of synthesis by (1), but the intersection $S^{n-1} = \overline{D} \cap \mathbb{R}^n \setminus D$ is not of synthesis.

(4) $E \subseteq \widehat{G}$ such that $\partial(E)$ is a Ditkin set, then E is a Ditkin set for $L^1(G)$. In particular, if $\partial(E)$ is countable, then E is a Ditkin set.

(5) Translates of sets of synthesis (Ditkin sets) are sets of synthesis (Ditkin sets).

(6) Let $\Gamma, \Gamma_1, \ldots, \Gamma_n$ be closed subgroups of $\widehat{G}$ such that $\Gamma_j \subseteq \Gamma$ and Γ_j is relatively open in Γ. Then, for any $\gamma_1, \ldots, \gamma_n \in \widehat{G}$, the set $\Gamma \setminus \bigcup_{j=1}^n \gamma_j \Gamma_j$ is a Ditkin set.

Malliavin's Theorem

Let G be a locally compact abelian group. If G is compact (equivalently, if $\widehat{G} = \Delta(L^1(G))$ is discrete), then spectral synthesis holds for $L^1(G)$, since $\emptyset$ is a Ditkin set.

Theorem (Malliavin's Theorem)

Spectral synthesis holds for $L^1(G)$ (if and) only if G is compact.

P. Malliavin, *Impossibilité de la synthèse spectrale sur les groupes abeliens non compact*, Inst. Hautes Et. Sci. Paris. **2** (1959), 61-68.

A more constructive proof than Malliavin's was given by Varopoulos, using tensor product methods:

N.Th. Varopoulos, *Tensor algebras and harmonic analysis*, Acta Math. **119** (1967), 57-111.

Steps of the Proof

(1) Let Γ be a closed subgroup of $\widehat{G}$ and

$$H = \{x \in G : \gamma(x) = 1 \text{ for all } \gamma \in \Gamma\}.$$

Let E be a closed subset of Γ and suppose that E is a set of synthesis for $L^1(G/H)$. Then E is a set of synthesis for $L^1(G)$.

(2) If $\mathbb{T} = \Delta(\ell^1(\mathbb{Z}))$ contains a set which is not of synthesis for $\ell^1(\mathbb{Z})$, then $\mathbb{R}$ contains a nonspectral set for $L^1(\mathbb{R})$.

Eevery locally compact abelian group contains an open subgroup H of the form $H = \mathbb{R}^n \times K$, where K is compact and $n \in \mathbb{N}_0$. Therefore (1) and (2) imply

(3) If spectral synthesis does not hold for every infinite discrete abelian group, then it does not hold for every noncompact locally compact abelian group.