Contractive homomorphisms from Fourier algebras: not new

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The Main Problem

Let G and H be locally compact groups.

Problem

Suppose that θ is a homomorphism from the Fourier algebras $\mathbb{A}(G)$ into the Fourier-Stieltjes algebra $\mathbb{B}(H)$. Describe θ.

Outline

(Contractive) homomorphisms $L^1(G) \to M(H)$

Suppose that $\phi : G \to M(H)$ is a uniformly bounded, weak*-continuous homomorphism.
Define

$$\phi^t(f)(t) := \langle \phi(t), f \rangle \quad (f \in \mathcal{C}_0(H),\ t \in G)\,.$$

Then $\phi^t : \mathcal{C}_0(H) \to \mathcal{C}^b(G)$ is a bounded linear map.
Define $\phi^{tt} : M(G) \to M(H)$ as follows.

$$\langle \phi^{tt}(\mu), f \rangle := \int_G \phi^t(f) \mathrm{d}\mu\,.$$

Then ϕ^{tt} is a bounded homomorphism.

A contractive homomorphism $L^1(G) \to M(H)$

Suppose that $\phi : G \to H$ is a continuous group homomorphism.
Define

$$\phi^t(f) := f \circ \phi \quad (f \in \mathcal{C}_0(H)) .$$

Then $\phi^t : \mathcal{C}_0(H) \to \mathcal{C}^b(G)$ is a contractive linear map.
Define $\phi^{tt} : M(G) \to M(H)$ as follows.

$$\langle \phi^{tt}(\mu), f \rangle := \int_G \phi^t(f) \mathrm{d}\mu .$$

Then ϕ^{tt} is a contractive homomorphism.

A more complicated one

Suppose that K is a compact supgroup of H that commutes with $\phi(G)$ i.e.

$$K\phi(G) \subseteq \phi(G)K\,.$$

Then we can define $\phi_K^{tt} : M(G) \to M(H)$ as follows.

$$\phi_K^{tt}(\mu) := \phi^{tt}(\mu) * m_K\,;$$

where m_K is the normalized Haar measure on K.
Then ϕ_K^{tt} is a contractive homomorphism.

If in addition we have a "nice" character $\rho : K \to \mathbb{T}$, we could modify

$$\phi_{K,\rho}^{tt}(\mu) := \phi^{tt}(\mu) * (\rho m_K)\,.$$

A further complication

If L is a normal supgroup of G such that

$$\phi(s) * (\rho m_K) = \rho m_K \quad (s \in L)\,,$$

i.e. $\psi(s) = \psi(1_G)$ for all $s \in L$, where $\psi := \phi^{tt}_{K,\rho}$.

Consider $\bar{\psi} : G/L \to M(H)$.

Define $\bar{\psi}^{tt} : M(G/L) \to M(H)$.

A more concrete example

Take $\Omega_0 \subseteq \Omega$ be closed subgroups of $\mathbb{T} \times H$ with

- Ω_0 is compact and normal in Ω, and
- $\pi_H : \Omega_0 \to H$ is injective.

Set $K := \pi_H(\Omega_0)$ and set $\rho := \pi_{\mathbb{T}} \circ (\pi_H|_{\Omega_0})^{-1}$.
Then

1. $\pi_H : \Omega \to H$ is a homomorphism,
2. K is a compact subgroup of H,
3. $\pi_H(\Omega)$ commutes with K, and
4. ρ is a "nice" character on K.

Thus we have a contractive homomorphism $\Phi : M(\Omega) \to M(H)$ as above.

Moreover,

- $\Phi(\Omega_0) = \Phi(1_\Omega)$.

Hence, a contractive homomorphism $\tilde{\Phi} : M(\Omega/\Omega_0) \to M(H)$.

Greenleaf's theorem

Let G and H be locally compact groups. Every contractive homomorphism $L^1(G) \to M(H)$ has the form

$$\tilde{\Phi} \circ \phi^{tt}$$

where

1. $\tilde{\Phi} : M(\Omega/\Omega_0) \to M(H)$ as above, and
2. $\phi : G \to \Omega/\Omega_0$ is a continuous epimorphism.

Homomorphisms from $\mathbb{A}(G)$.

- Suppose that $\theta : \mathbb{A}(G) \to \mathbb{B}(H)$ is a homomorphism.
- For each $t \in H$, either $\theta(f)(t) = 0 \; \forall f \in \mathbb{A}(G)$.
- Or, $f \mapsto \theta(f)(t)$ is a character of $\mathbb{A}(G)$.
- So that $\exists\, \tau(t) \in G$, $\theta(f)(t) = f(\tau(t))$ for all $f \in \mathbb{A}(G)$.
- Thus, $\exists$ an open subset Ω of H and a continuous map $\tau : \Omega \to G$ such that

$$\theta(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in \Omega \\ 0 & \text{if } t \in H \setminus \Omega \end{cases}, \qquad (\forall f \in \mathbb{A}(G)).$$

- As a consequence, $\theta : \mathbb{A}(G) \to \mathbb{B}(H)$ is automatically bounded.

Homomorphisms from $\mathbb{A}(G)$ (cont.)

- Conversely, given a map $\tau : \Omega \to G$, where $\Omega \subseteq H$. Define

$$\theta_\tau(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in \Omega \\ 0 & \text{if } t \in H \setminus \Omega \end{cases}, \qquad (\forall f \in \mathbb{A}(G)).$$

- Then $\theta_\tau : \mathbb{A}(G) \to \ell^\infty(H)$ is a homomorphism.
- Where $\ell^\infty(H)$ is the algebra of bounded functions on H.

Question

For which τ, does $\theta_\tau(\mathbb{A}(G)) \subseteq \mathbb{B}(H)$?

A reduction lemma

Let $\theta : \mathbb{A}(G) \to \mathbb{B}(H)$ be a homomorphism.
Then θ is induced by some continuous map $\tau : \Omega \to G$.
The formula

$$\varphi(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in \Omega \\ 0 & \text{if } t \in H \setminus \Omega \end{cases}$$

makes sense even if $f \in \mathbb{B}(G_d)$.
In fact, φ is a homomorphism from $\mathbb{A}(G_d)$ into $\mathbb{B}(H_d)$ with $\|\varphi\| \leq \|\theta\|$.

Proof of the reduction

It suffices to show that for $u = \sum_{i=1}^{m} \alpha_i \delta_{a_i}$ and $v = \sum_{i=1}^{m} \beta_i \delta_{b_i}$ in $c_{00}(G_0)$ with $\sum_{i=1}^{m} |\alpha_i|^2 = \sum_{i=1}^{m} |\beta_i|^2 = 1$ we have

$$\left| \sum_{k=1}^{n} \gamma_k \varphi(u * \check{v})(x_k) \right| \leq \|\theta\| \qquad (1)$$

for every finite systems $(x_k) \subseteq H$ and $(\gamma_k) \subset \mathbb{C}$ with $\| \sum_{k=1}^{n} \gamma_k \omega_{H_d}(x_k) \| \leq 1$.

The left hand side of (1) is

$$\left| \sum_{k=1}^{n} \gamma_k \varphi(u * \check{v})(x_k) \right| = \left| \sum_{x_k \in \Omega} \gamma_k (u * \check{v})(\tau(x_k)) \right|$$

$$= \left| \sum_{x_k \in \Omega} \sum_{i,j=1}^{m} \gamma_k \alpha_i \beta_j \delta_{a_i b_j^{-1}}(\tau(x_k)) \right|$$

Take a measurable set V to be chosen.
Consider $f = \sum_{i=1}^m \alpha_i \chi_{a_i V}$ and $g = \sum_{i=1}^m \beta_i \chi_{b_i V}$ in $L^2(G)$ both of L^2-norm $\sqrt{|V|}$.
So, $f * \check{g} \in \mathrm{A}(G)$ with norm at most $|V|$. Therefore,

$$\|\theta(f * \check{g})\| \leq \|\theta\| \, |V| \, .$$

Thus

$$\|\theta\| \, |V| \geq \left| \sum_{k=1}^n \gamma_k \theta(f * \check{g})(x_k) \right| = \left| \sum_{x_k \in \Omega} \gamma_k (f * \check{g})(\tau(x_k)) \right|$$

$$= \left| \sum_{x_k \in \Omega} \sum_{i,j=1}^m \gamma_k \alpha_i \beta_j \, \left| a_i V \cap \tau(x_k) b_j V \right| \right|$$

$$= \left| \sum_{x_k \in \Omega} \sum_{i,j=1}^m \gamma_k \alpha_i \beta_j \delta_{a_i b_j^{-1}}(\tau(x_k)) \right| \cdot |V| \, .$$

A reduction question

Let $\theta : \mathrm{A}(G) \to \mathrm{B}(H)$ be a homomorphism.
Then θ is induced by some continuous map $\tau : \Omega \to G$.
The formula

$$\varphi(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in \Omega \\ 0 & \text{if } t \in H \setminus \Omega \end{cases}$$

makes sense even if $f \in \mathrm{B}(G_d)$.

Is φ is a homomorphism from $\mathrm{B}(G_d)$ into $\mathrm{B}(H_d)$ with $\|\varphi\| \leq \|\theta\|$?

Anti-Affine Maps

- Suppose that C is an open coset of G.
- An anti-affine map $\tau : C \to H$ is a continuous map satisfying that

$$\tau(rs^{-1}t) = \tau(t)\tau(s)^{-1}\tau(r) \quad (r, s, t \in C).$$

- An anti-affine map $\tau : C \to H$ is a translation of a group anti-homomorphism:
  1. fix $s_0 \in C$, then $s_0^{-1}C$ is an open subgroup of G;
  2. the map $s \mapsto \tau(s_0)^{-1}\tau(s_0 s)$, $s_0^{-1}C \to H$ is a continuous group anti-homomorphism.

Isometric Isomorphisms

Theorem (Walter)

Let $\theta : \mathrm{A}(G) \to \mathrm{A}(H)$ be an isometric isomorphism. Then there exists an either affine or anti-affine homeomorphism τ from H onto G such that $\theta(f) = f \circ \tau$ ($\forall f \in \mathrm{A}(G)$).

Theorem (Walter)

Let $\theta : \mathrm{B}(G) \to \mathrm{B}(H)$ be an isometric isomorphism. Then there exists an either affine or anti-affine homeomorphism τ from H onto G such that $\theta(f) = f \circ \tau$ ($\forall f \in \mathrm{B}(G)$).

Contractive Homomorphisms from $\mathbb{A}(G)$ into $\mathbb{B}(H)$

Theorem

Suppose that $\theta : \mathbb{A}(G) \to \mathbb{B}(H)$ is a contractive homomorphism. Then there exist an open coset C and an either affine or anti-affine map $\tau : C \to G$ such that

$$\theta(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in C \\ 0 & \text{if } t \in H \setminus C \end{cases} \qquad (\forall f \in \mathbb{A}(G)).$$

Proof

We may assume that G and H have discrete topologies.

Suppose that θ is induced by $\tau : \Omega \to G$ where $\Omega \subseteq H$.

By composing with translations by elements of G and H, we suppose that $1_H \in \Omega$ and $\tau(1_H) = 1_G$.

Now, if $f \in \mathbb{A}(G)$ is positive definite, then

$$\theta(f)(1_H) = f(1_G) = \|f\| \geq \|\theta_\tau(f)\|,$$

and so $\theta(f)$ is also positive definite.

Thus if $t \in \Omega$, then

$$\theta(f)(t^{-1}) = \overline{\theta(f)(t)} = \overline{f(\tau(t))} = f(\tau(t)^{-1});$$

and so $t^{-1} \in \Omega$ and $\tau(t)^{-1} = \tau(t^{-1})$.

$\theta^* : \mathrm{W}^*(H) \to \mathrm{VN}(G)$ is a positive linear operator with $\theta^*(1) = 1$.
Let $t, s \in \Omega$ and $\alpha, \beta \in \mathbb{C}$ be arbitrary.
Set

$$a := \alpha\omega_H(t) + \beta\omega_H(s) + \overline{\alpha}\omega_H(t^{-1}) + \overline{\beta}\omega_H(s^{-1}).$$

Then $a = a^* \in \mathrm{W}^*(H)$.
Kadison's generalized Schwarz inequality: $\theta^*(a^2) \geq \theta^*(a)^2$.
We see that

$$\begin{aligned}\theta^*(a^2) = \mathrm{Re}\Big\{&2\left(|\alpha|^2 + |\beta|^2\right) + \alpha^2\ \theta^*[\omega_H(t^2)] + \beta^2\ \theta^*[\omega_H(s^2)]\\ &+ 2\alpha\beta\ \theta^*[\omega_H(ts) + \omega_H(st)]\\ &+ 2\alpha\overline{\beta}\ \theta^*[\omega_H(ts^{-1}) + \omega_H(s^{-1}t)]\Big\} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\theta^*(a)^2 = \mathrm{Re}\Big\{&2\left(|\alpha|^2 + |\beta|^2\right) + \alpha^2\lambda_G(\tau(t)^2) + \beta^2\lambda_G(\tau(s)^2)\\ &+ 2\alpha\beta[\lambda_G(\tau(t)\tau(s)) + \lambda_G(\tau(s)\tau(t))]\\ &+ 2\alpha\overline{\beta}[\lambda_G(\tau(t)\tau(s)^{-1}) + \lambda_G(\tau(s)^{-1}\tau(t))]\Big\}.\end{aligned}$$

It follows that

$$\theta^*[\omega_H(ts)] + \theta^*[\omega_H(st)] = \lambda_G[\tau(t)\tau(s)] + \lambda_G[\tau(s)\tau(t)].$$

Since $\lambda_G(G) = \{\lambda_G(x) : \ x \in G\}$ is linearly independent, and

$$\theta^*\circ\omega_H(x) = \lambda_G\circ\tau(x) \text{ if } x \in \Omega \quad \text{and} \quad \theta^*\circ\omega_H(x) = 0 \text{ if } x \in H\backslash\Omega.$$

we see that $ts, st \in \Omega$ and the ordered pair

$$\{\theta^*[\omega_H(ts)], \theta^*[\omega_H(st)]\} = \{\lambda_G[\tau(ts)], \lambda_G[\tau(st)]\}$$

is a permutation of $\{\lambda_G[\tau(t)\tau(s)], \lambda_G[\tau(s)\tau(t)]\}$.

Consequences

Corollary

Suppose that $\theta : \mathbb{A}(G) \to \mathbb{B}(H)$ is an injective and contractive homomorphism. Then θ is necessarily isometric.

Corollary

Suppose that $\theta : \mathbb{A}(G) \to \mathbb{A}(H)$ is a contractive isomorphism. Then there exists an either affine or anti-affine homeomorphism τ from H onto G such that $\theta(f) = f \circ \tau$ $(\forall f \in \mathbb{A}(G))$.

Contractive Homomorphism $\theta : \mathbb{A}(G) \to \mathbb{A}(H)$

Must be the composition of a diagram of the form

$$\mathbb{A}(G) \xrightarrow{L_{u_0}} \mathbb{A}(G) \xrightarrow{\pi} \mathbb{A}(G_0) \xrightarrow{\varphi}$$

$$\longrightarrow \mathbb{A}(\Omega/K) \xrightarrow{\iota} \mathbb{A}(\Omega) \xrightarrow{\rho} \mathbb{A}(H) \xrightarrow{L_{r_0}} \mathbb{A}(H);$$

where

- G_0 is a closed subgroup of G, Ω is an open subgroup of H, and K is a compact normal subgroup of Ω;
- L_{u_0} and L_{r_0} are (left) translations,
- π is the restriction to G_0 map, induced by $G_0 \hookrightarrow G$,
- φ is an isometric isomorphism,
- ι is the isometric monomorphism, induced by $\Omega \twoheadrightarrow \Omega/K$,
- ρ is the natural inclusion.

Contractive Isomorphisms of Fourier-Stieltjes Algebras

Let G and H be locally compact groups.

Theorem

Let $\theta : \mathbb{B}(G) \to \mathbb{B}(H)$ be an isomorphism such that $\theta|_{\mathbb{A}(G)}$ is contractive. Then there exists an either affine or anti-affine homeomorphism τ from H onto G such that $\theta(f) = f \circ \tau$ ($\forall f \in \mathbb{B}(G)$).

Corollary

In particular, θ is isometric and maps $\mathbb{A}(G)$ onto $\mathbb{A}(H)$.

A partial answer to the reduction question

Let $\theta : \mathrm{A}(G) \to \mathrm{B}(H)$ be a contractive homomorphism.
Says θ is induced by $\tau : \Omega \to G$.
The formula

$$\varphi(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in \Omega \\ 0 & \text{if } t \in H \setminus \Omega \end{cases}$$

gives a contractive homomorphism from $\mathrm{B}(G_d)$ into $\mathrm{B}(H_d)$!