HERZ-SCHUR MULTIPLIERS

IVAN G. TODOROV

Contents

1. Introduction

The purpose of these notes is to develop the basics of the theory of Herz-Schur multipliers. This notion was formally introduced in 1985 in [5] and developed by U. Haagerup and his collaborators, as well as by a number of other researchers, in the following decades. The literature on the subject is vast, and its applications – far reaching. A major driving force behind these developments were the connections with approximation properties of

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operator algebras. In these notes, we will not discuss this side of the subject, and will only briefly mention how Herz-Schur multipliers are used to define approximation properties of the Fourier algebra in Section 7. Instead, we focus on the development of the core material on multipliers on locally compact groups and various specific classes of interest. These notes formed the basis of a series of lectures at the programme "Harmonic analysis, Banach and operator algebras" at the Fields Institute in March-April 2014; due to time limitations, some aspects of the subject, such as that of Littlewood multipliers, are not included here.

2. Preliminaries

2.1. **Operator spaces.** We refer the reader to the monographs [8], [41], [42] for background in Operator Space Theory. In this section, we fix notation and include some results that will be used in the sequel. If $\mathcal{X}$ is a vector space, we denote as customary by $M_n(\mathcal{X})$ the vector space of all n by n matrices with entries in $\mathcal{X}$. If $\mathcal{Y}$ is another vector space and $\varphi : \mathcal{X} \to \mathcal{Y}$ is a linear map, we let $\varphi^{(n)} : M_n(\mathcal{X}) \to M_n(\mathcal{Y})$ be the map given by $\varphi^{(n)}((x_{i,j})) = (\varphi(x_{i,j}))$; thus, if we identify $M_n(\mathcal{X})$ and $M_n(\mathcal{Y})$ with $\mathcal{X} \otimes M_n$ and $\mathcal{Y} \otimes M_n$, respectively, then $\varphi^{(n)} = \varphi \otimes \mathrm{id}$.

If $\mathcal{H}$ is a Hilbert space and $(e_i)_{i\in I}$ is a fixed basis, we associate to every element $x \in \mathcal{B}(\mathcal{H})$ its matrix $(x_{i,j})_{i,j\in I}$. Here, $x_{i,j} = (xe_j, e_i), i, j \in I$. More generally, if $\mathcal{X}$ is an operator space then every element of the spacial norm closed tensor product $\mathcal{X} \otimes_{\min} \mathcal{K}(\mathcal{H})$ can be identified with a matrix $(x_{i,j})_{i,j\in I}$, but this time with $x_{i,j}$ being elements of $\mathcal{X}$. If $\varphi : \mathcal{X} \to \mathcal{Y}$ is a completely bounded linear map then there exists a (unique) bounded map $\varphi\otimes\mathrm{id} : \mathcal{X}\otimes_{\min}\mathcal{K} \to \mathcal{Y}\otimes_{\min}\mathcal{K}$ such that $\varphi\otimes\mathrm{id}((x_{i,j})) = (\varphi(x_{i,j}))$. If, moreover, $\mathcal{X}$ and $\mathcal{Y}$ are dual operator spaces and φ is weak*-continuous then there exists a (unique) weak* continuous bounded map $\tilde{\varphi} : \mathcal{X}\bar{\otimes}\mathcal{B}(\mathcal{H}) \to \mathcal{Y}\bar{\otimes}\mathcal{B}(\mathcal{H})$ such that $\tilde{\varphi}((x_{i,j})) = (\varphi(x_{i,j}))$ for every $(x_{i,j})_{i,j\in I} \in \mathcal{X}\bar{\otimes}\mathcal{B}(\mathcal{H})$. Here, $\bar{\otimes}$ denotes the weak* spacial tensor product. The map $\tilde{\varphi}$ will still be denoted by $\varphi \otimes \mathrm{id}$.

We next include the statement of two fundamental theorems in Operator Space Theory. The first one is Stinespring's Dilation Theorem:

Theorem 2.1. *Let $\mathcal{A}$ be a C^*-algebra and $\Phi : \mathcal{A} \to \mathcal{B}(H)$ be a completely positive map. There exist a Hilbert space K, a non-degenerate *-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$ and a bounded operator $V : H \to K$ such that*

$$\Phi(a) = V^*\pi(a)V, \quad a \in \mathcal{A}.$$

The second is the Haagerup-Paulsen-Wittstock Factorisation Theorem.

Theorem 2.2. *Let $\mathcal{A}$ be a C^*-algebra and $\Phi : \mathcal{A} \to \mathcal{B}(H)$ be a completely bounded map. There exist a Hilbert space K, a non-degenerate *-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$ and bounded operators $V, W : H \to K$ such* *that*

$$\Phi(a) = W^*\pi(a)V, \quad a \in \mathcal{A}.$$

Moreover, V and W can be chosen so that $\|\Phi\|_{\rm cb} = \|V\|\|W\|$.

We next include some results of R. R. Smith [47] and F. Pop, A. Sinclair and R. R. Smith [43] that will be useful in the sequel. Let H be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C*-algebra, and $\mathcal{X} \subseteq \mathcal{B}(H)$ be an operator space such that $\mathcal{A}\mathcal{X}\mathcal{A} \subseteq \mathcal{X}$; such an $\mathcal{X}$ is called an $\mathcal{A}$-bimodule. Since the C*-algebra $\mathcal{K}(H)$ of all compact operators on H is an ideal in $\mathcal{B}(H)$, it is an $\mathcal{A}$-bimodule for every C*-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$.

Let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{X}$ be an operator space that is an $\mathcal{A}$-bimodule. We call $\mathcal{A}$ *matricially norming* for $\mathcal{X}$ [43] if, for every $n \in \mathbb{N}$ and every $X \in M_n(\mathcal{X})$, we have that

$$\|X\| = \sup\{\|CXD\| : C = (c_1, \dots, c_n), D = (d_1, \dots, d_n)^t, \|C\|, \|D\| \le 1\}.$$

Theorem 2.3. *Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{X}$ be an $\mathcal{A}$-bimodule and $\Phi : \mathcal{X} \to \mathcal{X}$ be an $\mathcal{A}$-bimodular map. If Φ is bounded and $\mathcal{A}$ is matricially norming for $\mathcal{X}$ then Φ is completely bounded with $\|\Phi\|_{\rm cb} = \|\Phi\|$.*

Proof. We have that

$$\begin{aligned} & \|\Phi^{(n)}\| = \sup\{\|\Phi^{(n)}(X)\| : X \in M_n(\mathcal{X}) \text{ a contraction}\} \\ = \ & \sup\{\|C^*\Phi^{(n)}(X)D\| : X \in M_n(\mathcal{X}), C, D \in M_{n,1}(\mathcal{A}) \text{ contractions}\} \\ = \ & \sup\{\|\Phi(C^*XD)\| : X \in M_n(\mathcal{X}), C, D \in M_{n,1}(\mathcal{A}) \text{ contractions}\} \\ \le \ & \|\Phi\|. \end{aligned}$$

□

Theorem 2.4. *Let H be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C*-subalgebra with a cyclic vector. If $\mathcal{X} \subseteq \mathcal{B}(H)$ is an $\mathcal{A}$-bimodule then $\mathcal{A}$ is matricially norming for $\mathcal{X}$.*

Proof. Let $\xi \in H$ be a vector with $\overline{\mathcal{A}\xi} = H$ and let $X = (x_{i,j}) \in M_n(\mathcal{X})$ be an operator matrix with $\|X\| > 1$. Then there exist vectors $\xi' = (\xi_1, \dots, \xi_n)$ and $\eta' = (\eta_1, \dots, \eta_n)$ of norm strictly less than 1 such that $|(X\xi', \eta')| > 1$. Since $\overline{\mathcal{A}\xi} = H$, there exist elements $a_i, b_i \in \mathcal{A}$ such that $a_i\xi$ (resp. $b_i\xi$) is as close to η_i (resp. ξ_i) so that the vectors $\xi'' = (a_1\xi, \dots, a_n\xi)$ and $\eta'' = (b_1\xi, \dots, b_n\xi)$ still have norm strictly less than 1 and the inequality

$$|(X\xi'', \eta'')| > 1 \tag{1}$$

still holds. Let $a = \sum_{i=1}^n a_i^* a_i$ and $b = \sum_{i=1}^n b_i^* b_i$. We assume first that a and b are invertible. Let $\tilde{\xi} = b^{1/2}\xi$, $\tilde{\eta} = a^{1/2}\eta$, $c_i = a_i a^{-1/2}$ and $d_i = b_i b^{-1/2}$. Then we have that $c_i\tilde{\eta} = a_i\eta$ and $d_i\tilde{\xi} = b_i\xi$, $i = 1, \dots, n$, and, by (1), that

$$\left| \left(\sum_{i,j} c_i^* x_{i,j} d_j \tilde{\xi}, \tilde{\eta} \right) \right| > 1. \tag{2}$$

Moreover,

$$\|\tilde{\xi}\|^2 = (b^{1/2}\xi, b^{1/2}\xi) = (b\xi, \xi) = \sum_{i=1}^{n} \|b_i\xi\|^2 < 1,$$

and, similarly, $\|\tilde{\eta}\|^2 < 1$. It follows that $\|\sum_{i,j=1}^{n} c_i^* x_{i,j} d_j\| > 1$. On the other hand, the operator $\sum_{i,j=1}^{n} c_i^* x_{i,j} d_j$ is equal to the product $C(x_{i,j})D$, where $C = (c_1^*, \dots, c_n^*)$ and $D = (d_1, \dots, d_n)^t$. We have that

$$\|C\|^2 = \|\sum_{i=1}^{n} c_i^* c_i\| = \|\sum_{i=1}^{n} a^{-1/2} a_i^* a_i a^{-1/2}\| = \|I\| = 1$$

and, similarly, $\|D\| = 1$. Thus, we have that $\|C(x_{i,j})D\| > 1$. This completes the proof in the case both a and b are invertible.

In case a or b is not invertible, we consider, instead of the vectors ξ' and η', the vectors $(a_1\xi, \dots, a_n\xi, \epsilon\xi) \in H^{n+1}$ and $(b_1\xi, \dots, b_n\xi, \epsilon\xi) \in H^{n+1}$, and replace the operator matrix $X \in M_n(\mathcal{X})$ with the matrix $X \oplus 0 \in M_{n+1}(\mathcal{X})$. The corresponding operators $a = \epsilon^2 I + \sum_{i=1}^{n} a_i^* a_i$ and $b = \epsilon^2 I + \sum_{i=1}^{n} b_i^* b_i$ are now invertible and the proof proceeds as before. $\square$

Theorem 2.3 and 2.4 have the following consequence, which was first established by R. R. Smith in [47].

Theorem 2.5 (**R. R. Smith**). *Let H be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^*-subalgebra with a cyclic vector and $\mathcal{X} \subseteq \mathcal{B}(H)$ be an $\mathcal{A}$-bimodule. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}(H)$ is an $\mathcal{A}$-bimodular bounded linear map. Then Φ is completely bounded and $\|\Phi\|_{\rm cb} = \|\Phi\|$.*

2.2. **Harmonic analysis.** Throughout these notes, G will denote a locally compact group. For technical simplicity, we will assume throughout that G is second countable. If $E, F \subseteq G$, we let as usual $E^{-1} = \{s^{-1} : s \in E\}$, $EF = \{st : s \in E, t \in F\}$ and $E^n = \{s_1 \cdots s_n : s_i \in E, i = 1, \dots, n\}$ $(n \in \mathbb{N})$. Left Haar measure on G will be denoted by m, and it will be assumed to have total mass 1 if G is compact. Integration along m with respect to the variable s will be written ds. We write $L^p(G)$ for the corresponding Lebesgue space, for $1 \le p \le \infty$, and $M(G)$ for the space of all regular bounded Borel measures on G. The Riesz Representation Theorem identifies $M(G)$ with the Banach space dual of the space $C_0(G)$ of all continuous functions on G vanishing at infinity; the duality here is given by

$$\langle f, \mu \rangle = \int_G f(s) d\mu(s), \quad f \in C_0(G), \mu \in M(G).$$

Note that $M(G)$ is an involutive Banach algebra with respect to the convolution product $*$ defined through the relation

$$\langle f, \mu * \nu \rangle = \int_{G \times G} f(st) d\mu(s) d\nu(t), \quad f \in C_0(G), \mu, \nu \in M(G),$$

and the involution given by

$$\mu^*(E) = \overline{\mu(E^{-1})}, \quad \mu \in M(G), E \text{ a Borel subset of } G.$$

The space $L^1(G)$ can, by virtue of the Radon-Nikodym Theorem, be regarded as the closed ideal of all absolutely continuous with respect to m measures in $M(G)$. Note that the inherited convolution product on $L^1(G)$ turns it into an approximately unital involutive Banach algebra. The involution of $L^1(G)$ is given by

$$f^*(s) = \Delta(s)^{-1}\overline{f(s^{-1})}, \quad s \in G, f \in L^1(G).$$

Here, and in the sequel, Δ denotes the modular function of G, defined by the property

$$m(Es) = \Delta(s)m(E), \quad s \in G, E \text{ a Borel subset of } G.$$

Given a complex function f on G, we let

$$\check{f}(s) = f(s^{-1}), \quad \tilde{f}(s) = \overline{f(s^{-1})}, \quad s \in G.$$

If H is a Hilbert space, we denote by $\mathcal{U}(H)$ the group of all unitary operators acting on H. A *unitary representation* of G is a homomorphism $\pi : G \to \mathcal{U}(H)$, continuous in the strong (equivalently, the weak) operator topology. We often write $H_\pi = H$ to designate the dependence of H on π. Given such π, there exists a non-degenerate *-representation of $L^1(G)$ (which we will denote with the same symbol) such that

$$\pi(f) = \int_G f(s)\pi(s)ds, \quad f \in L^1(G),$$

in the norm topology of $\mathcal{B}(H_\pi)$. Two unitary representations π_1, π_2 of G are called *equivalent* if there exists a unitary operator $U \in \mathcal{B}(H_{\pi_1}, H_{\pi_2})$ such that $U\pi_1(s)U^* = \pi_2(s)$, $s \in G$. The set of all equivalence classes of irreducible unitary representations of G is denoted by $\hat{G}$ and called the *spectrum* of G. We think of $\hat{G}$ as a complete family of inequivalent irreducible representations of G.

A *coefficient* of π is a function on G of the form $s \to (\pi(s)\xi, \eta)$, where $\xi, \eta \in H$. The *Fourier-Stieltjes algebra* of G is the collection of all coefficients of unitary representations of G; it is clear that $B(G)$ is contained in the algebra $C^{\rm b}(G)$ of all bounded continuous functions on G. It is not difficult to see that $B(G)$ is an algebra with respect to pointwise addition and multiplication. It is moreover a Banach algebra with respect to the norm

$$\|u\| = \inf\{\|\xi\|\|\eta\| : u(\cdot) = (\pi(\cdot)\xi, \eta)\},$$

where the infimum is taken over all unitary representations π and all vectors ξ and η with the designated property.

For $s \in G$, let $\lambda_s \in \mathcal{U}(L^2(G))$ be given by $\lambda_s f(t) = f(s^{-1}t)$, $t \in G$, $f \in L^2(G)$. The map $\lambda : G \to \mathcal{U}(L^2(G))$ sending s to λ_s is a representation of G, called the *left regular representation*. The corresponding representation of $L^1(G)$ is faithful (and non-degenerate). The *Fourier algebra* $A(G)$ of G

is the collection of all coefficients of λ; it is a closed ideal of $B(G)$ and the norm on $A(G)$ is given by

$$\|u\| = \inf\{\|\xi\|\|\eta\| : u(s) = (\lambda_s\xi, \eta), s \in G, \xi, \eta \in L^2(G)\}.$$

Through the pivotal work of P. Eymard [10], $A(G)$ is a (commutative) semi-simple regular Banach algebra with spectrum G. We note that $A(G) \subseteq C_0(G)$ and $\|u\|_\infty \leq \|u\|$ for every $u \in A(G)$. Moreover, $B(G) \cap C_c(G) \subseteq A(G)$ (here $C_c(G)$ stands for the space of all continuous functions on G with compact support).

We denote by $C^*(G)$ the *C*-algebra* of G; this is the enveloping C*-algebra of $L^1(G)$, that is, the completion of $L^1(G)$ with respect to the norm

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ a unitary representation of } G\}$$

(note that the supremum on the right hand side is finite since, for every representation π of G and every $f \in L^1(G)$, we have $\|\pi(f)\| \leq \|f\|_1$). The C*-algebra $C^*(G)$ is characterised by the following universal property: for every unitary representation π of G, there exists a unique non-degenerate representation $\tilde{\pi}$ of $C^*(G)$ such that $\tilde{\pi}(f) = \pi(f)$ for every $f \in L^1(G)$. In the future, we will not use a different notation for $\tilde{\pi}$ and simply denote it by π. Note that $L^1(G)$ can be considered in a natural fashion as a norm dense *-subalgebra of $C^*(G)$.

The *reduced C*-algebra* $C_r^*(G)$ of G is the closure, in the operator norm, of the image $\lambda(L^1(G))$ inside $\mathcal{B}(L^2(G))$, while the *group von Neumann algebra* of G is the weak* (equivalently, the weak, or the strong, operator topology) closure of $C_r^*(G)$.

The Banach space dual of $C^*(G)$ can be isometrically identified with $B(G)$ *via* the formula

$$\langle f, u\rangle = \int_G f(s)u(s)ds, \quad f \in L^1(G), u \in B(G)\,. \tag{3}$$

In a similar fashion, the (unique) Banach space predual of $\mathrm{VN}(G)$ can be isometrically identified with $A(G)$; in addition to the formula (3) (where u is taken from $A(G)$), the duality is described by the formulas

$$\langle u, \lambda_s\rangle = u(s), \quad s \in G, u \in A(G)\,.$$

For a given $u \in A(G)$ and $T \in \mathrm{VN}(G)$, the functional on $A(G)$ given by $v \to \langle uv, T\rangle$, $v \in A(G)$, is bounded and thus there exists a (unique) element $u \cdot T \in \mathrm{VN}(G)$ such that

$$\langle v, u \cdot T\rangle = \langle uv, T\rangle, \quad v \in A(G).$$

Note that $\|u \cdot T\| \leq \|u\|\|T\|$. The map

$$A(G) \times \mathrm{VN}(G) \to \mathrm{VN}(G), \quad (u, T) \to u \cdot T,$$

is easily seen to define the structure of a Banach $A(G)$-module on $\mathrm{VN}(G)$. It can be shown that in fact $\mathrm{VN}(G)$ is an operator $A(G)$-module when equipped with this action; moreover, for each $u \in A(G)$, the map $T \to u \cdot T$ is weak* continuous.

The positive linear functionals on $C^*(G)$ correspond to positive definite functions from $B(G)$. A function $u \in C(G)$ (where $C(G)$ is the space of all continuous functions on G) is called *positive definite* if the matrix $(u(s_i s_j^{-1}))_{i,j=1}^n$ is positive for every choice $s_1, \dots, s_n$ of elements of G. Equivalently, u is positive definite if, viewed as an element of $L^\infty(G)$, it defines a positive linear functional on $L^1(G)$, that is, if

$$\int_G u(s)(f * f^*)(s)ds \geq 0, \quad f \in L^1(G).$$

We denote by $P(G)$ the collection of all continuous positive definite functions on G; it is easy to see that $P(G) \subseteq C^{\mathrm{b}}(G)$ and that if $u \in P(G)$ then $\|u\|_\infty = u(e)$.

Using GNS theory, one can show that $P(G) \subseteq B(G)$. More precisely, a function $u \in B(G)$ is positive definite if and only if there exists a representation π of G and a vector $\xi \in H_\pi$ (cyclic for π) such that $u(s) = (\pi(s)\xi, \xi)$, $s \in G$. We then say that u is a *positive coefficient* of π.

Let N_λ be the kernel of the left regular representation λ of $C^*(G)$. Clearly, $C_r^*(G) = C^*(G)/N_\lambda$, up to a *-isomorphism. The group G is called *amenable* if $N_\lambda = \{0\}$; in this case, the C*-algebras $C^*(G)$ and $C_r^*(G)$ are *-isomorphic. We note that amenability is usually defined by requiring the existence of a left invariant mean on $L^\infty(G)$. A further equivalent formulation of amenability can be derived as follows. Given two families $\mathcal{S}$ and $\mathcal{T}$ of representations of G (or, equivalently, of $C^*(G)$), say that $\mathcal{S}$ is weakly contained in $\mathcal{T}$ if $\cap_{\pi\in\mathcal{T}} \ker\pi \subseteq \cap_{\pi\in\mathcal{S}} \ker\pi$. Every positive linear functional on $C_r^*(G)$ gives rise, via composition with the corresponding quotient map, to a positive linear functional on $C^*(G)$ and can hence be identified with an element of $P(G)$. It was shown by J. M. G. Fell [11] that the elements of $P(G)$ obtained in this way are precisely the positive coefficients of the unitary representations of G weakly contained in λ. More precisely, we have the following facts.

Proposition 2.6. *Let $u \in P(G)$. The following are equivalent:*

(i) The formula

$$\lambda(f) \to \int_G u(s)f(s)ds, \quad f \in L^1(G),$$

defines a positive linear functional on $C_r^*(G)$;

(ii) The GNS representation corresponding to u is weakly contained in λ;

(iii) The function u is the limit, in the topology of uniform convergence on compacts, of functions of the form $f * \tilde{f}$, $f \in L^2(G)$.

We note that the functions of the form $f * \tilde{f}$ are precisely the positive coefficients of the left regular representation of G:

$$(\lambda_s(f), f) = \int_G f(s^{-1}t)f(t)dt = (f * \tilde{f})(s), \quad s \in G.$$

The set of functions satisfying the equivalent conditions of Proposition 2.6 will be denoted by $P_\lambda(G)$, and its linear span in $B(G)$, by $B_\lambda(G)$. Thus, the space $B_\lambda(G)$ corresponds in a canonical fashion to the Banach space dual of $C_r^*(G)$. We have the following additional characterisation of $B_\lambda(G)$:

Proposition 2.7. *The following are equivalent, for a function $u : G \to \mathbb{C}$:*
(i) $u \in B_\lambda(G)$;
(ii) the formula

$$\lambda(f) \to \int_G u(s)f(s)ds, \quad f \in L^1(G),$$

defines a bounded linear functional on $C_r^*(G)$;

From Proposition 2.7 one can easily derive that the group G is amenable if and only if the constant function 1 can be approximated, uniformly on compact sets, by elements of $A(G)$. Equivalently, G is amenable if and only if $A(G)$ possesses a bounded approximate identity [34].

Since $B(G) = C^*(G)^*$, we can equip $B(G)$ with the operator space structure arising from this duality. Similarly, $A(G)$ (resp. $B_\lambda(G)$), being the predual of $\mathrm{VN}(G)$ (resp. the dual of $C_r^*(G)$), can be equipped with a canonical operator space structure. Throughout these notes, any reference to $A(G)$, $B(G)$ and $B_\lambda(G)$ as operator spaces utilises the structures just introduced.

If G and H are locally compact groups, we denote as customary by $G \times H$ the direct product of G and H equipped with the product topology. We have that $\mathrm{VN}(G \times H) = \mathrm{VN}(G) \bar{\otimes} \mathrm{VN}(H)$. It follows that, up to a complete isometry, $A(G \times H) = A(G) \hat{\otimes} A(H)$, where $\hat{\otimes}$ denotes the operator projective tensor product.

3. The spaces $MA(G)$ and $M^{\mathrm{cb}}A(G)$

In this section, we define Herz-Schur multipliers and establish some of their basic properties. We follow closely [5], where Herz-Schur multipliers were first introduced and studied.

Definition 3.1. *A function $u : G \to \mathbb{C}$ is called a* multiplier *of $A(G)$ if $uv \in A(G)$ for every $v \in A(G)$.*

We denote the set of all multipliers of $A(G)$ by $MA(G)$. Clearly, $MA(G)$ is an algebra with respect to pointwise addition and multiplication. We note that if $u \in MA(G)$ then u is continuous; indeed, given $s \in G$, choose a compact neighbourhood K of s and let $v \in A(G)$ be a function with $v|_K = 1$. Then $uv|_K = u|_K$, and since uv is continuous, it follows that u is continuous at s.

If $u \in MA(G)$, let $m_u : A(G) \to A(G)$ be the map given by $m_u(v) = uv$, $v \in A(G)$. We note that the map m_u satisfies the relation $m_u(vw) = vm_u(w)$ for all $v, w \in A(G)$.

Proposition 3.2. *If $u \in MA(G)$ then the map m_u is bounded.*

Proof. The (linear) map m_u is defined on a Banach space; in order to show that m_u is bounded, it suffices, by the Closed Graph Theorem, to show that m_u has closed graph. Let therefore $(u_k)_{k\in\mathbb{N}} \subseteq A(G)$ be a null sequence such that $uu_k \to v$ for some $v \in A(G)$. Then $\|u_k\|_\infty \to_{k\to\infty} 0$ and $\|uu_k - v\|_\infty \to_{k\to\infty} 0$. Thus,

$$v(s) = \lim_{k\to\infty} u(s)u_k(s) = 0, \quad s \in G;$$

in other words, $v = 0$ as an element of $A(G)$. □

Exercise 3.3. *Suppose that $T : A(G) \to A(G)$ is a linear map such that $T(vw) = vT(w)$, $v, w \in A(G)$. Then there exists $u \in MA(G)$ such that $T = m_u$.*

For $u \in MA(G)$, we set $\|u\|_{\mathrm{m}} \stackrel{def}{=} \|m_u\|$.

Remark 3.4. *We have that $B(G) \subseteq MA(G)$. Moreover, if $u \in B(G)$ then $\|u\|_{\mathrm{m}} \leq \|u\|_{B(G)}$.*

Proof. Since $A(G)$ is an ideal of $B(G)$, we have that $B(G) \subseteq MA(G)$. Moreover,

$$\|m_u\| = \sup\{\|uv\| : v \in A(G), \|v\| \leq 1\} \leq \|u\|_{B(G)}.$$

□

Definition 3.5. *An element $v \in MA(G)$ is called a* completely bounded multiplier *of $A(G)$ if the map m_v is completely bounded.*

Let $M^{\mathrm{cb}}A(G)$ be set of all completely bounded multipliers of $A(G)$. Since $m_{uv} = m_u m_v$ (for $u, v \in MA(G)$), we have that $M^{\mathrm{cb}}A(G)$ is a subalgebra of $MA(G)$. Set $\|u\|_{\mathrm{cbm}} = \|m_u\|_{\mathrm{cb}}$ (where $u \in M^{\mathrm{cb}}A(G)$); then $M^{\mathrm{cb}}A(G)$ is a Banach algebra with respect to $\|\cdot\|_{\mathrm{cbm}}$.

If $u \in MA(G)$, the dual map m_u^* of m_u acts on $\mathrm{VN}(G)$; we will denote it by S_u. If $s \in G$ and $v \in A(G)$ then

$$\langle v, S_u(\lambda_s)\rangle = \langle m_u(v), \lambda_s\rangle = \langle uv, \lambda_s\rangle = u(s)v(s) = \langle v, u(s)\lambda_s\rangle.$$

This shows that $S_u(\lambda_s) = u(s)\lambda_s$, $s \in G$. In particular, it follows that

$$|v(s)| = \|v(s)\lambda_s\| = \|S_v(\lambda_s)\| \leq \|v\|_{\mathrm{m}}, \quad s \in G; \tag{4}$$

and thus the elements of $MA(G)$ are bounded functions. The above argument also proves a part of the following theorem.

Theorem 3.6. *Let $u : G \to \mathbb{C}$ be a bounded continuous function. The following are equivalent:*

(i) $u \in MA(G)$*;*

(ii) There exists a (unique) bounded weak continuous linear map T on $\mathrm{VN}(G)$ such that $T(\lambda_s) = u(s)\lambda_s$, $s \in G$;*

(iii) There exists a bounded linear map R on $C_r^*(G)$ such that

$$R(\lambda(f)) = \lambda(uf), \quad f \in L^1(G);$$

(iv) $uv \in B_\lambda(G)$ *for every* $v \in B_\lambda(G)$.

Proof. (i)$\Rightarrow$(ii) follows from the argument before the statement of the theorem, by taking $T = S_u$.

(ii)$\Rightarrow$(iii) We claim that the restriction R of T to $C_r^*(G)$ satisfies the given relations. Indeed, letting $f \in L^1(G)$, we have that, in the topology of the norm, $\lambda(f) = \int_G f(s)\lambda_s ds$. Since T is norm continuous,

$$T(\lambda(f)) = \int_G f(s)T(\lambda_s)ds = \int_G f(s)u(s)\lambda_s ds. \tag{5}$$

Since u is bounded (see (4)), $uf \in L^1(G)$ and (5) shows that $T(\lambda(f)) = \lambda(uf)$.

(iii)$\Rightarrow$(iv) For $v \in B_\lambda(G)$ and $f \in L^1(G)$, we have

$$\left|\int_G u(s)v(s)f(s)ds\right| = |\langle \lambda(uf), v\rangle| \leq \|R\|\|\lambda(f)\|\|v\|_{B_\lambda(G)}.$$

It follows that the map

$$\lambda(f) \to \int_G u(s)v(s)f(s)ds, \quad f \in L^1(G),$$

extends to a bounded linear functional on $C_r^*(G)$ of norm not exceeding $\|R\|\|v\|_{B_\lambda(G)}$. By Proposition 2.7, $uv \in B_\lambda(G)$.

(iv)$\Rightarrow$(i) An application of the Closed Graph Theorem as in the proof of Proposition 3.2 shows that the map $v \to uv$ on $B_\lambda(G)$ is bounded. Suppose that $v \in B(G) \cap C_c(G)$; then $uv \in B_\lambda(G) \cap C_c(G) \subseteq A(G)$. Since $\overline{B(G) \cap C_c(G)} = A(G)$ [10], it follows that $uA(G) \subseteq A(G)$. $\square$

Remark 3.7. *Let $u \in L^\infty(G)$. The following are equivalent:*

(i) u is equivalent (with respect to the Haar measure) to a function from $MA(G)$;

(ii) there exists $C > 0$ such that

$$\|\lambda(uf)\| \leq C\|\lambda(f)\|, \quad f \in L^1(G).$$

Proof. (i)$\Rightarrow$(ii) follows from Theorem 3.6 and the fact that if $u \sim v$ then $\lambda(uf) = \lambda(vf)$ for every $f \in L^1(G)$.

(ii)$\Rightarrow$(i) Let $v \in B_\lambda(G)$ and $\omega : \lambda(L^1(G)) \to \mathbb{C}$ be the functional given by

$$\omega(\lambda(f)) = \int_G ufvdm, \quad f \in L^1(G).$$

Then

$$|\omega(\lambda(f))| \leq C\|v\|_{B_\lambda(G)}\|\lambda(f)\|, \quad f \in L^1(G).$$

Thus, there exists $w \in B_\lambda(G)$ such that

$$\omega(\lambda(f)) = \int_G wfdm, \quad f \in L^1(G).$$

It follows that $uv = w$ almost everywhere. Since such a function w exists for every choice of $v \in B_\lambda(G)$, we conclude that u agrees almost everywhere with a continuous function. The statement in (i) now follows from Theorem 3.6. $\square$

Note that $u \in M^{\rm cb}A(G)$ if and only if the map T, or that map R, from Theorem 3.6 are in fact completely bounded.

We next characterise the elements of $M^{\rm cb}A(G)$ within $MA(G)$. If H is another locally compact group and $u : G \to \mathbb{C}$, we write $u \times 1$ for the function defined on $G \times H$ by $u \times 1(s,t) = u(s)$, $s \in G$, $t \in H$. To underline the dependence of this function on H, we write $u \times 1_H$. Recall that $SU(n)$ denotes the special unitary group in dimension n, that is, the group (under multiplication) of all n by n unitary matrices with determinant 1.

Theorem 3.8. *Let $u \in MA(G)$. The following are equivalent:*

(i) $u \in M^{\rm cb}A(G)$;

(ii) $u \times 1 \in MA(G \times H)$ *for every locally compact group H;*

(iii) $u \times 1 \in MA(G \times SU(2))$.

Moreover, if these conditions are fulfilled then

$$\|u\|_{\rm cbm} = \sup_{H \ l.c.g.} \|u \times 1_H\|_{\rm m} = \|u \times 1_{SU(2)}\|_{\rm m}.$$

Proof. (i)⇒(ii) By assumption, the map $S_u : {\rm VN}(G) \to {\rm VN}(G)$ is completely bounded and weak* continuous. Let $\mathcal{H} = L^2(H)$. The map $S_u \otimes {\rm id} : {\rm VN}(G)\bar{\otimes}\mathcal{B}(\mathcal{H}) \to {\rm VN}(G)\bar{\otimes}\mathcal{B}(\mathcal{H})$ is bounded and weak* continuous with $\|S_u \otimes {\rm id}\,\| \leq \|S_u\|_{\rm cb}$. We have that $(S_u \otimes {\rm id})(T \otimes S) = S_u(T) \otimes S$, $T \in {\rm VN}(G)$, $S \in \mathcal{B}(\mathcal{H})$. In particular, if $s \in G$ and $t \in H$ then

$$(S_u \otimes {\rm id})(\lambda_s \otimes \lambda_t) = S_u(\lambda_s) \otimes \lambda_t = (u \times 1)(s,t)\lambda_{(s,t)}.$$

By Theorem 3.6, $u \times 1 \in MA(G \times H)$.

(ii)⇒(iii) is trivial.

(iii)⇒(i) The group $SU(2)$ is compact; by the Peter-Weyl Theorem,

$${\rm VN}(SU(2)) \cong \oplus^{\ell^\infty}_{\pi \in \hat{SU(2)}} \mathcal{B}(H_\pi)$$

as von Neumann algebras. It is well-known that for every $n \in \mathbb{N}$ there exists a unique equivalence class of irreducible unitary representations of $SU(2)$ whose underlying Hilbert space has dimension n. Thus, ${\rm VN}(SU(2)) \cong \oplus_{n=1}^\infty M_n$. It follows that

$$\text{(6)} \qquad {\rm VN}(G)\bar{\otimes}\,{\rm VN}(SU(2)) \cong \oplus_{n=1}^\infty {\rm VN}(G) \otimes M_n.$$

For $s \in G$ and $t \in SU(2)$, we have that $S_{u\times 1}(\lambda_s \otimes \lambda_t) = S_u(\lambda_s) \otimes \lambda_t$. By linearity and weak* continuity, we have that $S_{u\times 1} = S_u \otimes {\rm id}\,|_{SU(2)}$. By (6), if $T = \oplus_{n=1}^\infty T_n \in {\rm VN}(G)\bar{\otimes}\,{\rm VN}(SU(2))$ then $S_{u\times 1}(T) = \oplus_{n=1}^\infty S_u^{(n)}(T_n)$. Thus, $\|S_u^{(n)}(T_n)\| \leq \|u\times 1\|_{\rm m}\|T_n\|$, $n \in \mathbb{N}$. Since $\{T_n : T \in {\rm VN}(G)\bar{\otimes}\,{\rm VN}(SU(2))\} = {\rm VN}(G)\otimes M_n$, we conclude that $\|S_u^{(n)}\| \leq \|u\times 1\|_{\rm m}$, and (i) is established. □

Corollary 3.9. *We have $B(G) \subseteq M^{\rm cb}A(G)$. Moreover, if $u \in B(G)$ then $\|u\|_{\rm cbm} \leq \|u\|_{B(G)}$.*

Proof. Let $u \in B(G)$ and H be any locally compact group. Then $u \times 1_H \in B(G \times H)$; indeed, if $\pi : G \to \mathcal{B}(H_\pi)$ is a unitary representation of G then $\pi \otimes 1 : G \times H \to \mathcal{B}(H_\pi)$ given by $\pi \otimes 1(s,t) = \pi(s)$ is a unitary representation of $G \times H$. It follows that $u \times 1_H \in B(G \times H)$; moreover, $\|u \times 1_H\|_{B(G\times H)} \leq \|u\|_{B(G)}$. By Remark 3.4, $u \times 1_H \in MA(G \times H)$ and

$$\|u \times 1_H\|_{\mathrm{m}} \leq \|u \times 1_H\|_{B(G\times H)} \leq \|u\|_{B(G)}.$$

It follows by Theorem 3.8 that $u \in M^{\mathrm{cb}}A(G)$ and $\|u\|_{\mathrm{cbm}} \leq \|u\|_{B(G)}$. □

It follows from Corollary 3.9 that

$$B(G) \subseteq M^{\mathrm{cb}}A(G) \subseteq MA(G).$$

It was shown by V. Losert [35] that G is amenable if and only if $B(G) = MA(G)$.

The following simple observation will be useful in the sequel.

Proposition 3.10. *If $u \in A(G)$ then $S_u(T) = u \cdot T$ for every $T \in \mathrm{VN}(G)$.*

Proof. If $s \in G$ and $v \in A(G)$ then

$$\langle S_u(\lambda_s), v\rangle = \langle u(s)\lambda_s, v\rangle = u(s)v(s)\lambda_s = \langle \lambda_s, uv\rangle = \langle u \cdot \lambda_s, v\rangle.$$

The claim follows by linearity and weak* continuity. □

3.1. **The case of commutative groups.** In this subsection, we follow the exposition of [44]. We assume throughout that G is abelian. We briefly recall some basic facts about Fourier theory on G. Let $\Gamma = \hat{G}$ be the dual group of G. If $f \in L^1(\Gamma)$, let $\hat{f} : G \to \mathbb{C}$ be its Fourier transform, namely, the function

$$\hat{f}(s) = \int_\Gamma f(\gamma)\overline{\gamma(s)}d\gamma, \quad s \in G.$$

We also set $\mathcal{F}(f) = \hat{f}$, $f \in L^1(\Gamma)$. Then

$$\|\mathcal{F}(f)\|_2 = \|f\|_2, \quad f \in L^1(\Gamma) \cap L^2(\Gamma),$$

and thus $\mathcal{F}$ extends to an isometry (denoted again by $\mathcal{F}$) from $L^2(\Gamma)$ *onto* $L^2(G)$. We often write $\hat{f} = \mathcal{F}(f)$ for elements f of $L^2(\Gamma)$. Note that, if $f, g \in L^1(\Gamma)$, then $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$. This implies that if $f, g \in L^2(\Gamma)$ are such that $\hat{f}, \hat{g} \in L^1(G)$ then $\widehat{fg} = \hat{f} * \hat{g}$. These observations form the base for the following fact.

Proposition 3.11. *We have that $A(G) = \{\hat{f} : f \in L^1(\Gamma)\}$. Moreover, the map $f \to \hat{f}$ is an isometric homomorphism of $L^1(\Gamma)$ onto $A(G)$.*

Fourier transform gives a useful insight into the C*-algebra and the von Neumann algebra of Γ. Indeed, let $L^\infty(G)$ act on $L^2(G)$ via multiplication; more precisely, consider the algebra

$$\mathcal{D}_G = \{M_\varphi : \varphi \in L^\infty(G)\},$$

where $M_\varphi \in \mathcal{B}(L^2(G))$ is given by $M_\varphi f = \varphi f$, $f \in L^2(G)$. Let also

$$\mathcal{C}_G = \{M_\varphi : \varphi \in C_0(G)\},$$

A straightforward calculation shows that

$$\mathcal{F}\lambda(f)\mathcal{F}^* = M_{\hat{f}}, \quad f \in L^1(\Gamma). \tag{7}$$

It follows that

$$\mathcal{F}\,\mathrm{VN}(\Gamma)\mathcal{F}^* = \mathcal{D}_G, \quad \mathcal{F}C_r^*(\Gamma)\mathcal{F}^* = \mathcal{C}_G.$$

Our next aim is to characterise similarly $B(G)$ and to show that $MA(G)$ coincides with it. First note that the Fourier transform can be extended to the algebra $M(G)$ of all Radon measures on Γ; for $\mu \in M(G)$, set

$$\hat{\mu}(s) = \int_\Gamma \overline{\gamma(s)} d\mu(\gamma), \quad s \in G.$$

Note that $\hat{\mu}$ is a continuous function on G with $\|\hat{\mu}\|_\infty \leq \|\mu\|$ (the latter norm being the total variation of μ).

The following is a classical result of Bochner's.

Theorem 3.12. *We have that*

$$P(G) = \{\hat{\mu} : \mu \in M(\Gamma), \text{ positive}\}.$$

Thus, $B(G) = \{\hat{\mu} : \mu \in M(\Gamma)\}$.

Proof. Let $u \in P(G)$, and assume, without loss of generality, that $u(e) = 1$. *Via* the identification of $B(G)$ with $C^*(G)^*$, the function u corresponds to a state ω_u of $C^*(G)$. The Cauchy-Schwarz inequality for positive linear functionals now implies that

$$|\omega_u(f)|^2 \leq \omega_u(f * \tilde{f}), \quad f \in L^1(G). \tag{8}$$

Fix $f \in L^1(G)$ and let $h = f * \tilde{f}$. Then a successive application of (8) shows that

$$|\omega_u(f)|^2 \leq (\|h^{2^n}\|_1)^{2^{-n}}.$$

Taking a limit, we obtain that

$$|\omega_u(f)|^2 \leq r(h),$$

where $r(h)$ is the spectral radius of h as an element of the Banach algebra $L^1(G)$. We have that $r(h) = \|\hat{h}\|_\infty$, and hence

$$|\omega_u(f)|^2 \leq \|\hat{h}\|_\infty.$$

It now follows that the map $\hat{f} \to \omega_u(f)$ is well-defined and bounded in the uniform norm.

On the other hand, an application of the Stone-Weierstrass Theorem shows that $A(\Gamma)$ is dense in $C_0(\Gamma)$ in $\|\cdot\|_\infty$. By the Riesz Representation Theorem, there exists a positive measure $\mu \in M(\Gamma)$ such that

$$\omega_u(f) = \int_\Gamma \hat{f} d\mu, \quad f \in L^1(G).$$

Thus,

$$\int_G f(s)u(s)ds = \omega_u(f) = \int_G \int_\Gamma f(s)\overline{\gamma(s)} d\mu(\gamma)ds, \quad f \in L^1(G).$$

It now follows that $u = \hat{\mu}$ almost everywhere. Since both u and $\hat{\mu}$ are continuous, we conclude that $u = \hat{\mu}$ everywhere.

Conversely, suppose that $\mu \in M(\Gamma)$ is a positive measure. For any choice $s_1, \dots, s_n$ of points in G, and any choice of scalars $\lambda_1, \dots, \lambda_n$, we have

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \hat{\mu}(s_i - s_j) &= \int_\Gamma \sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \overline{\gamma(s_i - s_j)} d\mu(\gamma) \\ &= \int_\Gamma \sum_{i,j=1}^{n} \overline{\lambda_i \gamma(s_i)} \lambda_j \gamma(s_j) d\mu(\gamma).\end{aligned}$$

Since $(\overline{\gamma(s_i)}\gamma(s_j))_{i,j=1}^n$ is a positive matrix for all $\gamma \in \Gamma$, we have that

$$\sum_{i,j=1}^{n} \lambda_i \overline{\gamma(s_i)} \overline{\lambda_j} \gamma(s_j) \geq 0$$

for all $\gamma \in \Gamma$. Since μ is positive, we conclude that

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} \hat{\mu}(s_i - s_j) \geq 0.$$

This shows that $\hat{\mu}$ is a positive definite function.

The second equality follows from the fact that $B(G)$ is the linear span of $P(G)$. ☐

Remark It can be shown that, if $\mu \in M(\Gamma)$, then $\|\hat{\mu}\|_{B(G)} = \|\mu\|$, where the latter denotes the total variation of μ.

We have the following alternative description of $B(G)$. Let $\mathcal{T}(\Gamma)$ be the linear space of all *trigonometric polynomials* on Γ, that is, the space of all functions $f : \Gamma \to \mathbb{C}$ of the form

$$f(\gamma) = \sum_{i=1}^{n} c_i \langle \gamma, s_i \rangle, \quad \gamma \in \Gamma, \tag{9}$$

where $s_i \in G$ and $c_i \in \mathbb{C}$, $i = 1, \dots, n$. We note that, equivalently, a trigonometric polynomial of the form (9) can be identified with the element $T_f \in \mathrm{VN}(G)$ given by

$$T_f = \sum_{i=1}^{n} c_i \lambda_{s_i}.$$

Proposition 3.13. *Let $u : G \to \mathbb{C}$ be a continuous function. The following are equivalent:*

(i) $u \in B(G)$ and $\|u\| \leq C$;

(ii) if f is a trigonometric polynomial of the form (9) then

$$\left| \sum_{i=1}^{n} c_i u(s_i) \right| \leq C \|f\|_\infty. \tag{10}$$

Proof. (ii)⇒(i) Let G_d be the group G equipped with the discrete topology. For $f \in \mathcal{T}(\Gamma)$, we have, in view of (7), that $\|T_f\| = \|f\|_\infty$. Thus, (10) implies that the linear map $\omega : T_f \to \sum_{i=1}^n c_i u(s_i)$ has the property $|\omega(T_f)| \le C\|T_f\|$, $f \in \mathcal{T}(\Gamma)$. Since $\{T_f : f \in \mathcal{T}(\Gamma)\}$ is dense in $C^*(G_d)$ in norm, the functional ω has an extension to a bounded linear functional on $C^*(G_d)$. Thus, there exists $v \in B(G_d)$ such that

$$\omega(T_f) = \langle T_f, v\rangle = \sum_{i=1}^n c_i v(s_i), \quad f \in \mathcal{T}(\Gamma).$$

It follows that $u = v$. However, by the Bochner-Eberlein Theorem, $B(G_d) \cap C(G) = B(G)$, and the proof is complete.

(i)⇒(ii) By virtue of the Bochner-Eberlein Theorem, $B(G_d) \cap C(G) = B(G)$, and hence $u \in B(G_d)$. The claim now follows from the fact that, if f is as in (9), then $\langle T_f, u\rangle = \sum_{i=1}^n c_i u(s_i)$. □

Theorem 3.14. *Suppose that $u : G \to \mathbb{C}$ is a function such that $uv \in B(G)$ for every $v \in A(G)$. Then $u \in B(G)$. In particular, $MA(G) = B(G)$.*

Moreover, if $u \in B(G)$ then $\|u\|_{\mathrm{m}} = \|u\|$.

Proof. One can easily show that u is continuous; moreover, a straightforward application of the Closed Graph Theorem (see the proof of Proposition 3.2) shows that the map $T : A(G) \to B(G)$ given by $T(v) = uv$, is bounded. Let $s_1, \dots, s_n \in G$, $c_1, \dots, c_n \in \mathbb{C}$, and $f = \sum_{i=1}^n c_i s_i \in \mathcal{T}(\Gamma)$ be the corresponding trigonometric polynomial on Γ. For a given $\epsilon > 0$, let $v \in A(G)$ be a function such that $v(s_i) = 1$, $i = 1, \dots, n$, and $\|v\| \le 1 + \epsilon$. Then $uv(s_i) = u(s_i)$, $i = 1, \dots, n$. Since $uv \in B(G)$, Theorem 3.12 gives an element $\mu \in M(\Gamma)$ such that $\hat{\mu} = uv$. Thus,

$$\begin{aligned} \left|\sum_{i=1}^n c_i u(s_i)\right| &= \left|\sum_{i=1}^n c_i \hat{\mu}(s_i)\right| \\ &= \left|\int_\Gamma \left(\sum_{i=1}^n c_i \overline{\gamma(s_i)}\right) d\mu(\gamma)\right| \le \|\mu\|\|f\|_\infty. \end{aligned}$$

By Proposition 3.13, $u \in B(G)$ and

$$\|u\| \le \|\mu\| = \|uv\|_{B(G)} \le \|T\|(1+\epsilon);$$

thus, $\|u\| \le \|T\|$. We have that $\|T\| = \|u\|_{\mathrm{m}}$ since the image of the map T is in $A(G)$. Thus, $\|u\| \le \|u\|_{\mathrm{m}}$; by Corollary 3.9, we have that $\|u\| = \|u\|_{\mathrm{m}}$. □

Corollary 3.15. *Let G be a locally compact abelian group. Then $M^{\mathrm{cb}}A(G) = B(G)$. Moreover, if $u \in B(G)$ then $\|u\| = \|u\|_{\mathrm{cbm}}$.*

4. Schur multipliers

This section is dedicated to a brief introduction to measurable Schur multipliers, which will be used in subsequent parts of the present text.

4.1. **ω-topology.** We fix for the whole section standard measure spaces (X,μ) and (Y,ν); by this we mean that there exist locally compact, metrisable, complete topologies on X and Y (called the *underlying topologies*), with respect to which μ and ν are regular Borel σ-finite measures.

By a *measurable rectangle* we will mean a subset of $X\times Y$ of the form $\alpha\times\beta$, where α and β are measurable. We denote by $\mu\times\nu$ the product measure (defined on the product σ-algebra on $X\times Y$, that is, on the σ-algebra generated by all measurable rectangles).

A subset $E\subseteq X\times Y$ will be called *marginally null* if there exist null sets $M\subseteq X$ and $N\subseteq Y$ such that $E\subseteq(M\times Y)\cup(X\times N)$. Every marginally null subset of $X\times Y$ is clearly a $\mu\times\nu$-null set. The converse is not true; for an example, consider the subset $\Delta=\{(x,x):x\in[0,1]\}$ of $[0,1]\times[0,1]$, where the unit interval $[0,1]$ is equipped with Lebesgue measure.

Two measurable sets $E,F\subseteq X\times Y$ will be called *marginally equivalent* if the symmetric difference of E and F is marginally null; in this case we write $E\cong F$. The sets E and F will be called *equivalent* if their symmetric difference is $\mu\times\nu$-null; in this case we write $E\sim F$. Similarly, for measurable functions $\varphi,\psi:X\times Y\to\mathbb{C}$, we write $\varphi\sim\psi$ (resp. $\varphi\cong\psi$) if the set $\{(x,y):\varphi(x,y)\neq\psi(x,y)\}$ is null (resp. marginally null). A measurable subset $\kappa\subseteq X\times Y$ is called *ω-open* if it is marginally equivalent to a subset of $X\times Y$ of the form $\cup_{i=1}^{\infty}\alpha_i\times\beta_i$, where $\alpha_i\subseteq X$ and $\beta_i\subseteq Y$ are measurable, $i\in\mathbb{N}$. The set κ will be called *ω-closed* if its complement κ^c is ω-open. The set of all ω-open sets is a *pseudo-topology*, that is, it is closed under taking countable unions and finite intersections.

Lemma 4.1 ([9]). *Suppose that the underlying topologies of X and Y are compact and the measures μ and ν are finite. Let κ be an ω-closed set, and γ_k, $k\in\mathbb{N}$, be ω-open subsets, of $X\times Y$, such that $\kappa\subseteq\cup_{k=1}^{\infty}\gamma_k$. For every $\epsilon>0$ there exist measurable sets $X_\epsilon\subseteq X$ and $Y_\epsilon\subseteq Y$ such that $\mu(X\setminus X_\epsilon)<\epsilon$, $\mu(Y\setminus Y_\epsilon)<\epsilon$ and the set $\kappa\cap(X_\epsilon\times Y_\epsilon)$ is contained in the union of finitely many of the sets γ_k, $k\in\mathbb{N}$.*

A function $h:X\times Y\to\mathbb{C}$ will be called *ω-continuous* if $h^{-1}(U)$ is ω-open for every open set $U\subseteq\mathbb{C}$. Let $C_\omega(X\times Y)$ be the set of all (marginal equivalence classes of) ω-continuous functions on $X\times Y$.

The following facts will be useful; their proofs are left as an exercise.

Proposition 4.2. *(i) The set $C_\omega(X\times Y)$ is an algebra with respect to pointwise addition and multiplication.*

(ii) If $\varphi,\psi\in C_\omega(X\times Y)$ and $\varphi\sim\psi$ then $\varphi\cong\psi$.

4.2. **The predual of $\mathcal{B}(H_1,H_2)$.** We let $H_1=L^2(X,\mu)$ and $H_2=L^2(Y,\nu)$. It is well-known that the dual Banach space of the space $\mathcal{C}_1(H_2,H_1)$ of all trace class operators from H_2 into H_1 is isometrically isomorphic to $\mathcal{B}(H_1,H_2)$, the duality being given by

$$\langle S,T\rangle=\mathrm{tr}(ST),\quad S\in\mathcal{C}_1(H_2,H_1),T\in\mathcal{B}(H_1,H_2),$$

where tr denotes the canonical trace on $\mathcal{C}_1(H_1)$. In this subsection we describe an identification of $\mathcal{C}_1(H_2, H_1)$ with a certain function space on $X \times Y$, which will be used in the rest of the section. Recall first that $\mathcal{C}_1(H_2, H_1)$ can be naturally identified with the projective tensor product $H_1 \hat{\otimes} H_2$ by identifying an elementary tensor $f \otimes g$, where $f \in H_1$ and $g \in H_2$, with the operator $T_{f\otimes g}$ of rank one given by

$$T_{f\otimes g}(h) = (h, \overline{g})f = \left(\int_Y h(y)g(y)d\nu(y)\right) f, \quad h \in H_2.$$

In this way, the operators of finite rank from H_2 into H_1 are identified with elements of the algebraic tensor product $L^2(X, \mu) \otimes L^2(Y, \nu$.

Lemma 4.3. *Suppose that $\sum_{j=1}^n f_j \otimes g_j = 0$ as an element of $L^2(X, \mu) \otimes L^2(Y, \nu)$. Then $\sum_{j=1}^n f_j(x)g_j(y) = 0$ for marginally almost all (x, y).*

Proof. Let $\psi(x, y) = \sum_{j=1}^n f_j(x)g_j(y)$, $(x, y) \in X \times Y$. The function ψ is well-defined up to a marginally null set. We first note that Reψ arises from the element

$$\frac{1}{2}\sum_{j=1}^n f_j \otimes g_j + \frac{1}{2}\sum_{j=1}^n \overline{f_j} \otimes \overline{g_j}$$

of $L^2(X, \mu) \otimes L^2(Y, \nu)$, which coincides with the zero element since both terms are zero. Similarly, Imψ arises from the element

$$\frac{1}{2i}\sum_{j=1}^n f_j \otimes g_j - \frac{1}{2i}\sum_{j=1}^n \overline{f_j} \otimes \overline{g_j}$$

of $L^2(X, \mu) \otimes L^2(Y, \nu)$ which is zero. If we show that Reψ and Imψ, viewed as functions, are equal to zero marginally almost everywhere, the lemma will be established. We may hence assume that the function ψ takes real values.

By Proposition 4.2, ψ is ω-continuous. Suppose that ψ is not marginally equivalent to the zero function; without loss of generality, assume that there exist $\delta > 0$ and a rectangle $\alpha \times \beta$ of finite non-zero measure such that $\psi(x, y) > \delta$ for all $(x, y) \in \alpha \times \beta$. But then

$$0 < \int_{\alpha\times\beta} \psi(x, y)d\mu(x)d\nu(y) = \sum_{j=1}^n (f_j, \chi_\alpha)(g_j, \chi_\beta) = 0,$$

a contradiction. □

For an element $u = \sum_{j=1}^n f_j \otimes g_j \in L^2(X, \mu) \otimes L^2(Y, \nu)$, we let ψ_u be the function on $X \times Y$ given by $\psi_u(x, y) = \sum_{j=1}^n f_j(x)g_j(y)$. By Lemma 4.3, ψ_u is well-defined, as an element of $C_\omega(X \times Y)$.

Lemma 4.4. *Let $\{u_n\}_{n=1}^\infty \in L^2(X, \mu) \otimes L^2(Y, \nu)$ be a sequence converging to zero in the projective tensor norm, and $\psi_n = \psi_{u_n}$. Then there exists a subsequence $\{n_k\}_{k=1}^\infty$ of natural numbers such that $\psi_{n_k} \to_{k\to\infty} 0$ marginally almost everywhere.*

Proof. We may assume that $u_n = \sum_{j=1}^{p_n} f_j^{(n)} \otimes g_j^{(n)}$, and

$$\sum_{j=1}^{p_n} \|f_j^{(n)}\|_2^2 \rightarrow_{n\rightarrow\infty} 0, \quad \sum_{j=1}^{p_n} \|g_j^{(n)}\|_2^2 \rightarrow_{n\rightarrow\infty} 0.$$

Thus,

$$\int_X \left(\sum_{j=1}^{p_n} |f_j^{(n)}(x)|^2 \right) d\mu(x) \rightarrow_{n\rightarrow\infty} 0$$

and hence there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\sum_{j=1}^{p_{n_k}} |f_j^{(n_k)}(x)|^2 \rightarrow_{n\rightarrow\infty} 0 \text{ almost everywhere.}$$

We may assume that, moreover,

$$\sum_{j=1}^{p_{n_k}} |g_j^{(n_k)}(y)|^2 \rightarrow_{n\rightarrow\infty} 0 \text{ almost everywhere.}$$

By the Cauchy-Schwarz inequality,

$$|\psi_{n_k}(x,y)|^2 \leq \sum_{j=1}^{p_{n_k}} |f_j^{(n_k)}(x)|^2 \sum_{j=1}^{p_{n_k}} |g_j^{(n_k)}(y)|^2 \longrightarrow_{k\rightarrow\infty} 0$$

marginally almost everywhere. □

Now let $u \in L^2(X,\mu)\hat{\otimes}L^2(Y,\nu)$, and suppose that $u = \sum_{j=1}^{\infty} f_j \otimes g_j$, where $\sum_{j=1}^{\infty} \|f_j\|_2^2 < \infty$ and $\sum_{j=1}^{\infty} \|g_j\|_2^2 < \infty$. Since $\sum_{j=1}^{\infty} \|f_j\|_2^2 < \infty$ we have that

$$\sum_{j=1}^{\infty} |f_j(x)|^2 < \infty \text{ almost everywhere on } X$$

and

$$\sum_{j=1}^{\infty} |g_j(x)|^2 < \infty \text{ almost everywhere on } Y.$$

By the Cauchy-Schwarz inequality, the sum $\sum_{j=1}^{\infty} f_j(x)g_j(y)$ is finite for marginally all (x,y). Let $\psi = \psi_u$ be the complex function defined marginally almost everywhere on $X \times Y$ by letting

$$\psi(x,y) = \sum_{j=1}^{\infty} f_j(x)g_j(y). \tag{11}$$

We note that the function $\psi(x,y)$ does not depend on the representation of u. To this end, suppose that $u = \sum_{j=1}^{\infty} \xi_j \otimes \eta_j$ is another representation of u and let $\phi(x,y) = \sum_{j=1}^{\infty} \xi_j(x)\eta_j(y)$. Set $u_n = \sum_{j=1}^{n} f_j \otimes g_j$, $v_n = \sum_{j=1}^{n} \xi_j \otimes \eta_j$, $\psi_n(x,y) = \sum_{j=1}^{n} f_j(x)g_j(y)$ and $\phi_n(x,y) = \sum_{j=1}^{n} \xi_j(x)\eta_j(y)$.

Thus, $\psi_n(x,y) \to \psi(x,y)$ and $\phi_n(x,y) \to \phi(x,y)$ marginally almost everywhere. We have that $\|u_n - v_n\|_\wedge \longrightarrow_{n\to\infty} 0$; by Lemma 4.4, there exists a subsequence $\{n_k\}_{k=1}^\infty$ of natural numbers such that $\psi_{n_k}(x,y) - \phi_{n_k}(x,y) \to 0$ marginally almost everywhere. Thus, $\psi(x,y) = \phi(x,y)$ marginally almost everywhere.

We now let $T(X,Y)$ be the space of all classes (with respect to marginal equivalence) of functions ψ_u, associated to elements $u \in L^2(X,\mu)\hat{\otimes}L^2(Y,\nu)$. We equip $T(X,Y)$ with the norm $\|\psi_u\|_\wedge \stackrel{def}{=} \|u\|_\wedge$. It is easy to note that, conversely, if $\psi : X \times Y \to \mathbb{C}$ is a function which admits a representation of the form (11), where $\sum_{j=1}^\infty \|f_j\|_2^2 < \infty$ and $\sum_{j=1}^\infty \|g_j\|_2^2 < \infty$, then $\psi = \psi_u$, where $u = \sum_{i=1}^\infty f_i \otimes g_i$.

If $u = \sum_{i=1}^\infty f_i \otimes g_i$, let $T_u : H_2 \to H_1$ be the nuclear operator given by

$$T_u(\eta)(x) = \sum_{i=1}^\infty (\eta, \overline{g_i}) f_i, \quad \eta \in H_2.$$

It is immediate that T_u is an integral operator with integral kernel ψ_u.

We note that if $k \in L^2(Y \times X)$, $T_k \in \mathcal{C}_2(H_1, H_2)$ is the corresponding Hilbert-Schmidt operator given by

$$T_k\xi(y) = \int_X k(y,x)\xi(x)d\mu(x), \quad y \in Y,$$

and if $u \in T(X,Y)$ then

$$\langle T_u, T_k\rangle = \int_{X\times Y} \psi_u(x,y)k(y,x)d\mu\times\nu(x,y). \tag{12}$$

Indeed, (12) can be verified first in the case T_ψ is an operator of rank one and then its validity follows by linearity and weak* continuity. If $u \in T(X,Y)$ and $T \in \mathcal{B}(H_1,H_2)$, we will often write $\langle u, T\rangle$ for $\langle T_u, T\rangle$.

Remark 4.5. *The map sending an element u of $L^2(X,\mu)\hat{\otimes}L^2(Y,\nu)$ to its corresponding class (with respect to marginal equivalence) of functions in $T(X,Y)$ is injective. That is, if $u_1, u_2 \in L^2(X,\mu)\hat{\otimes}L^2(Y,\nu)$ and $\psi_{u_1} \cong \psi_{u_2}$ then $T_{u_1} = T_{u_2}$.*

Proof. This is immediate from the fact that T_{u_1} and T_{u_2} are integral operators with integral kernels ψ_{u_1} and ψ_{u_2}, respectively. □

Henceforth, we identify the space of (marginal equivalence classes of) functions $T(X,Y)$ with the projective tensor product $L^2(X,\mu)\hat{\otimes}L^2(Y,\nu)$; we thus suppress the distinction between u and ψ_u and use the same symbol to denote them.

We note that equation (12) implies the following, which will be useful in the sequel: suppose that $\psi \in T(X,Y)$ and ψ' is a measurable function with $\psi' \sim \psi$. Then, clearly,

$$\int_{X\times Y} \psi(x,y)k(y,x)d\mu\times\nu(x,y) = \int_{X\times Y} \psi'(x,y)k(y,x)d\mu\times\nu(x,y),$$

for all $k \in L^2(Y \times X)$. It follows that the map

$$T_k \to \int_{X\times Y} \psi'(x,y)k(y,x)d\mu \times \nu(x,y)$$

is bounded in the operator norm, and hence there exists $\psi'' \in T(X,Y)$ such that

$$\langle T_{\psi''}, T_k\rangle = \int_{X\times Y} \psi'(x,y)k(y,x)d\mu \times \nu(x,y).$$

It now follows that $\psi'' \sim \psi'$, and thus $\psi'' \sim \psi$. Since both ψ'' and ψ are ω-continuous, we have by Proposition 4.2 that $\psi \cong \psi''$. Thus, the function ψ' is the integral kernel of the operator T_ψ. In other words, integral operators $T_{\psi'}$ can be defined unambiguously for any function ψ' that is equivalent, with respect to the product measure, to a function from $T(X,Y)$.

Proposition 4.6. *The inclusion $T(X,Y) \subseteq C_\omega(X \times Y)$ holds.*

Proof. We first establish the following

Claim. *If $f_n : X \times Y \to \mathbb{C}$ and $\phi_n : X \times Y \to \mathbb{R}^+$ are ω-continuous functions such that $\inf_n \phi_n(x,y) = 0$ for marginally almost all x, y, and if $f : X\times Y \to \mathbb{C}$ is a function with $|f(x,y)-f_n(x,y)| \leq \phi_n(x,y)$ for marginally almost all x, y, then f is ω-continuous.*

Proof of Claim. It is easy to reduce the statement to the case where f_n and f are real valued. In this case, however, for any $a \in \mathbb{R}$, up to a marginally null set,

$$f^{-1}((a,+\infty)) = \bigcup_{m,n=1}^{\infty} f_n^{-1}((a+\frac{1}{m},+\infty)) \cap \phi_n((0,\frac{1}{m})).$$

The claim now follows.

Let $h = \sum_{i=1}^\infty f_i \otimes g_i$, where $\sum_{i=1}^\infty \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|_2^2 < \infty$. Set $\phi_{n+1}(x,y) = \sum_{i=n}^\infty (|f_i(x)|^2 + |g_i(y)|^2)$, $n \in \mathbb{N}$. Then the functions ϕ_n are ω-continuous, $\inf_n \phi_n(x,y) = 0$ for marginally almost all x, y and if we let $h_n = \sum_{i=1}^n f_i \otimes g_i$ we see that $|h_n - h| \leq \phi_n$ up to a marginally null set, for each n. The statement is now immediate by the Claim. □

4.3. **The space $T(G)$.** Let G be a locally compact group. We write $T(G) = T(G,G)$. The map $P : T(G) \to A(G)$, given by

$$P(f \otimes g)(t) = \langle \lambda_t, f \otimes g\rangle = (\lambda_t f, \overline{g}) = \int_G f(t^{-1}s)g(s)ds = g * \check{f}(t) \tag{13}$$

is a contractive surjection, by the definition of $A(G)$. The next lemma will be useful later.

Lemma 4.7. *If $h \in T(G)$ then*

$$P(h)(t) = \int_G h(t^{-1}s, s)ds, \quad t \in G. \tag{14}$$

Proof. Identity (14) is a direct consequence of (13) if h is a finite sum of elementary tensors. Let $h = \sum_{i=1}^{\infty} f_i \otimes g_i \in T(G)$, where $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$, and let h_n be the nth partial sum of this series. By the continuity of P, $\|P(h_n) - P(h)\| \to 0$ in $A(G)$; since $\|\cdot\|_\infty$ is dominated by the norm of $A(G)$, we conclude that $P(h_n)(t) \to P(h)(t)$ for every $t \in G$.

By Lemma 4.4, there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ such that $h_{n_k} \to h$ marginally almost everywhere. It follows that, for every $t \in G$, one has $h_{n_k}(t^{-1}s, s) \to h(t^{-1}s, s)$ for almost all $s \in G$. By [36, (4.3)], the function $s \to \sum_{i=1}^{\infty} |f_i(t^{-1}s)||g_i(s)|$ is integrable, and hence an application of the Lebesgue Dominated Convergence Theorem shows that $\int_G h_{n_k}(t^{-1}s, s)ds \to_{k\to\infty} \int_G h(t^{-1}s, s)ds$, for every $t \in G$. The proof is complete. □

4.4. **The characterisation theorem.** If $h : X \times Y \to \mathbb{C}$ is a function then, by writing $h \in^{\mu\times\nu} T(X, Y)$, we will mean that h is equivalent, with respect to the measure $\mu \times \nu$, to a function that lies in $T(X, Y)$. If $h \in^{\mu\times\nu} T(X, Y)$ then there exists a unique, up to marginal equivalence, element h' of $T(X, Y)$ such that $h \sim h'$. Indeed, if $h \sim h'$ and $h \sim h''$, where $h', h'' \in T(X, Y)$, then $h' \sim h''$ and, by Propositions 4.6 and 4.2, $h' \cong h''$.

Definition 4.8. *A function $\varphi \in L^\infty(X \times Y)$ is called a* Schur multiplier *if $\varphi h \in^{\mu\times\nu} T(X, Y)$ for every $h \in T(X, Y)$.*

Let $\mathfrak{S}_{\mu,\nu}(X, Y)$ be the set of all Schur multipliers on $X \times Y$ with respect to a pair of fixed measures μ, ν. When the measures are understood from the context, we simply write $\mathfrak{S}(X, Y)$. We note that, strictly speaking, Schur multipliers are classes of functions with respect to almost everywhere equality.

If $\varphi \in \mathfrak{S}(X, Y)$, let $m_\varphi : T(X, Y) \to T(X, Y)$ be given by $m_\varphi h = \varphi h$. Note that, strictly speaking, $m_\varphi h$ is (defined to be) the (unique, up to marginal equivalence) function $h' \in T(X, Y)$ such that $h' \sim \varphi h$. We also note that if $\varphi, \psi \in \mathfrak{S}(X, Y)$ and $\varphi \sim \psi$ then $m_\varphi = m_\psi$. Thus, the map m_φ is independent of the representative φ we use to define it.

Proposition 4.9. *If $\varphi \in \mathfrak{S}(X, Y)$ then the operator m_φ on $T(X, Y)$ is bounded.*

Proof. We apply the Closed Graph Theorem. Suppose that $(h_k)_{k\in\mathbb{N}} \subseteq T(X, Y)$ is such that $\|h_k\|_\wedge \to 0$ and $\|\varphi h_k - h\|_\wedge \to 0$ for some $h \in T(X, Y)$. Let h'_k be the unique element from $T(X, Y)$ such that $\varphi h_k \sim h'_k$, $k \in \mathbb{N}$. Using Lemma 4.4, we may assume, after passing to subsequences, that $h_k \to 0$ and $h'_k \to h$ marginally almost everywhere. It follows that $\varphi h_k \to h$ almost everywhere, and hence $h = 0$ almost everywhere. Since h is ω-continuous, Proposition 4.2 implies that $h = 0$ marginally almost everywhere, and thus $h = 0$ as an element of $T(X, Y)$. □

If $\varphi \in \mathfrak{S}(X, Y)$, we write $\|\varphi\|_{\mathfrak{S}} = \|m_\varphi\|$. Our next aim is to give a characterisation of Schur multipliers; we follow the approach of [29]. For

$\varphi \in \mathfrak{S}(X,Y)$, we let $S_\varphi = m_\varphi^*$; thus, $S_\varphi : \mathcal{B}(H_1,H_2) \to \mathcal{B}(H_1,H_2)$ is a bounded weak* continuous map.

For $a \in L^\infty(X,\mu)$ let M_a be the operator on $L^2(X,\mu)$ defined by $M_a f = af$. Let $\mathcal{D}_X = \{M_a : a \in L^\infty(X,\mu)\}$; define $\mathcal{D}_Y$ analogously. For a function $\varphi : X\times Y \to \mathbb{C}$, let $\hat\varphi : Y\times X \to \mathbb{C}$ be the function given by $\hat\varphi(y,x) = \varphi(x,y)$.

Theorem 4.10. *Let $\varphi \in \mathfrak{S}(X,Y)$. Then S_φ is a weak* continuous completely bounded $\mathcal{D}_Y,\mathcal{D}_X$-module map and, if $k \in L^2(Y\times X)$, then $S_\varphi(T_k) = T_{\hat\varphi k}$.*

Conversely, if $\Phi : \mathcal{B}(H_1,H_2) \to \mathcal{B}(H_1,H_2)$ is a a weak continuous bounded $\mathcal{D}_Y,\mathcal{D}_X$-module map then there exists a unique $\varphi \in \mathfrak{S}(X,Y)$ such that $\Phi = S_\varphi$.*

Proof. Suppose that $\varphi \in \mathfrak{S}(X,Y)$. The fact that S_φ is a bounded weak* continuous map was observed after the proof of Proposition 4.9. Let $k \in L^2(Y\times X)$ and $h \in T(X,Y)$. Using (12), we have

$$\begin{aligned}\langle S_\varphi(T_k), T_h\rangle &= \langle T_k, m_\varphi(T_h)\rangle = \langle T_k, T_{\varphi h}\rangle \\ &= \int_{X\times Y} k(y,x)\varphi(x,y)h(x,y)d\mu\times\nu(x,y) = \langle T_{\hat\varphi k}, T_h\rangle.\end{aligned}$$

Thus, $S_\varphi(T_k) = T_{\hat\varphi k}$.

Now let $a \in L^\infty(X,\mu)$ and $b \in L^\infty(Y,\nu)$; for $k \in L^2(Y\times X)$ and $h \in T(X,Y)$ we have

$$\begin{aligned}\langle S_\varphi(M_b T_k M_a), T_h\rangle &= \int_{X\times Y} a(x)b(y)\varphi(x,y)k(y,x)h(x,y)d\mu\times\nu(x,y) \\ &= \langle S_\varphi(T_k), M_a T_h M_b\rangle = \langle M_b S_\varphi(T_k) M_a, T_h\rangle;\end{aligned}$$

thus, S_φ is a $\mathcal{D}_Y,\mathcal{D}_X$-module map.

It is easy to see that $\mathcal{D}_X$ and $\mathcal{D}_Y$ have cyclic vectors. By Theorem 2.5, S_φ is completely bounded.

The proof of the converse direction follows the lines of [29]. Suppose that $\Phi : \mathcal{B}(H_1,H_2) \to \mathcal{B}(H_1,H_2)$ is a a weak* continuous bounded $\mathcal{D}_Y,\mathcal{D}_X$-module map. By Theorem 2.5, Φ is completely bounded. By a well-known result of U. Haagerup's [17], there exists a bounded (row) operator $B = (M_{b_k})_{k\in\mathbb{N}} \in M_{1,\infty}(\mathcal{D}_Y)$ and a bounded (column) operator $A = (M_{a_k})_{k\in\mathbb{N}} \in M_{\infty,1}(\mathcal{D}_X)$ such that

$$\Phi(T) = \sum_{k=1}^\infty M_{b_k} T M_{a_k}, \quad T \in \mathcal{B}(H_1,H_2),$$

where the series converges in the weak* topology. We have that

$$C_1 = \operatorname*{esssup}_{x\in X} \sum_{k=1}^\infty |a_k(x)|^2 < \infty \text{ and } C_2 = \operatorname*{esssup}_{y\in Y} \sum_{k=1}^\infty |b_k(y)|^2 < \infty,$$

and hence the function

$$\varphi(x,y)=\sum_{k=1}^{\infty}a_k(x)b_k(y)$$

is well-defined up to a marginally null set. We show that $\varphi\in\mathfrak{S}(X,Y)$; let $h=\sum_{i=1}^{\infty}f_i\otimes g_i\in T(X,Y)$. Then

$$\varphi(x,y)h(x,y)=\sum_{k,i}a_k(x)f_i(x)b_k(y)g_i(y),\qquad \text{m.a.e..}$$

However,

$$\begin{aligned}\sum_{k,i}\|a_kf_i\|_2^2 &= \int_X\sum_{k,i}|a_k(x)f_i(x)|^2d\mu(x)\leq C_1\int_X\sum_i|f_i(x)|^2d\mu(x)\\ &= C_1\sum_{i=1}^{\infty}\|f_i\|_2^2;\end{aligned}$$

similarly,

$$\sum_{k,i}\|b_kg_i\|_2^2\leq C_2\sum_{i=1}^{\infty}\|g_i\|_2^2,$$

and we are done.

For $k\in L^2(Y\times X)$ we now have $\Phi(T_k)=S_\varphi(T_k)$; since both Φ and S_φ are bounded and weak*-continuous, we conclude by the weak* density of $\mathcal{C}_2(H_1,H_2)$ in $\mathcal{B}(H_1,H_2)$ that $\Phi=S_\varphi$. □

Theorem 4.10 and its proof show the following.

Corollary 4.11. *The map from $\mathfrak{S}(X,Y)$ into the space $CB^{w^*}_{\mathcal{D}_Y,\mathcal{D}_X}(\mathcal{B}(H_1,H_2))$ of all completely bounded weak* continuous $\mathcal{D}_Y,\mathcal{D}_X$-module maps, sending φ to S_φ, is a bijective isometry.*

Exercise 4.12 ([48]). *Show that the map from Corollary 4.11 is a complete isometry.*

In the sequel, we call φ the *symbol* of S_φ and equip $\mathfrak{S}(X,Y)$ with the operator space structure that makes the map $\varphi\to S_\varphi$ a complete isometry. By a well-known result of U. Haagerup's [17] (see also [1]), the space $CB^{w^*}_{\mathcal{D}_Y,\mathcal{D}_X}(\mathcal{B}(H_1,H_2))$ is completely isometric and weak* homeomorphic to the weak* Haagerup tensor product $\mathcal{D}_Y\otimes_{\mathrm{w^*h}}\mathcal{D}_X$ *via* the mapping sending an element $\sum_{k=1}^{\infty}B_k\otimes A_k\in\mathcal{D}_Y\otimes_{\mathrm{w^*h}}\mathcal{D}_X$ to the map $T\to\sum_{k=1}^{\infty}B_kTA_k$. Utilising the canonical isomorphism between $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$) and $L^\infty(X,\mu)$ (resp. $L^\infty(Y,\nu)$), we see that $\mathcal{D}_Y\otimes_{\mathrm{w^*h}}\mathcal{D}_X$ can be viewed as a space of (equivalence classes of) functions, and that it can be identified with $\mathfrak{S}(X,Y)$. We summarise this as a part of the theorem that follows.

Theorem 4.13. *Let $\varphi\in L^\infty(X\times Y)$. The following are equivalent:*

(i) $\varphi\in\mathfrak{S}(X,Y)$ *and* $\|\varphi\|_{\mathfrak{S}}\leq C$;

(ii) there exists sequences $(a_k)_{k=1}^{\infty} \subseteq L^{\infty}(X,\mu)$ *and* $(b_k)_{k=1}^{\infty} \subseteq L^{\infty}(Y,\nu)$ *with*

$$C_1 \stackrel{def}{=} \operatorname*{esssup}_{x\in X} \sum_{k=1}^{\infty} |a_k(x)|^2 \leq C \text{ and } C_2 \stackrel{def}{=} \operatorname*{esssup}_{y\in Y} \sum_{k=1}^{\infty} |b_k(y)|^2 \leq C,$$

such that

$$\varphi(x,y) = \sum_{k=1}^{\infty} a_k(x) b_k(y) \qquad a.e. \text{ on } X \times Y;$$

(iii) there exist a separable Hilbert space K *and weakly measurable functions* $a : X \to K$, $b : Y \to K$, *such that*

$$\operatorname*{esssup}_{x\in X} \|a(x)\| \leq \sqrt{C}, \quad \operatorname*{esssup}_{y\in Y} \|b(y)\| \leq \sqrt{C}$$

and

$$\varphi(x,y) = (a(x), b(y)), \qquad a.e. \text{ on } X \times Y;$$

(iv) $\|T_{\hat{\varphi}k}\| \leq C\|T_k\|$ *for all* $k \in L^2(Y \times X)$.

Proof. The equivalence (i)⇔(ii) was established in the proof of Theorem 4.10.

(iv)⇒(i) Let $h \in T(X,Y)$. The functional

$$T_k \to \int_{X\times Y} \varphi(x,y) k(y,x) h(x,y) d\mu \times \nu(x,y)$$

on $\mathcal{C}_2(H_1,H_2)$ is bounded in the operator norm, and has norm not exceeding C. It follows that $\varphi h \in T(X,Y)$ and $\|\varphi h\|_{\wedge} \leq C$. Thus, $\varphi \in \mathfrak{S}(X,Y)$ and $\|\varphi\|_{\mathfrak{S}} \leq C$.

(i)⇒(iv) follows from Theorem 4.10.

(ii)⇒(iii) Set $K = \ell^2$, $a(x) = (a_k(x))_{k=1}^{\infty}$ and $b(y) = (b_k(y))_{k=1}^{\infty}$.

(iii)⇒(ii) Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis of K and set $a_k(x) = (a(x), e_k)$, $b_k(y) = (e_k, b(y))$. Then

$$\sum_{k=1}^{\infty} |a_k(x)|^2 = \sum_{k=1}^{\infty} (a(x), e_k)(e_k, a(x)) = \|a(x)\|^2$$

and similarly for $b(y)$; thus the boundedness conditions follow. Similarly,

$$(a(x), b(y)) = \sum_{k=1}^{\infty} (a(x), e_k)(e_k, b(y)) = \sum_{k=1}^{\infty} a_k(x) b_k(y)$$

holds for almost all (x,y). □

Corollary 4.14. *Every element of* $\mathfrak{S}(X,Y)$ *is equivalent to a (unique) function from* $C_{\omega}(X \times Y)$.

Proof. By Theorem 4.13, every element of $\mathfrak{S}(X,Y)$ is equivalent, with respect to the product measure on $X \times Y$, to a function of the form

$$(x,y) \to \sum_{k=1}^{\infty} a_k(x) b_k(y),$$

where the sequences $(a_k)_{k=1}^{\infty} \subseteq L^{\infty}(X,\mu)$ and $(b_k)_{k=1}^{\infty} \subseteq L^{\infty}(Y,\nu)$ satisfy the conditions in Theorem 4.13 (ii). It is now easy to check that all such functions are ω-continuous. $\square$

An important subclass of Schur multipliers is formed by the positive ones. A Schur multiplier $\varphi \in \mathfrak{S}(X,X)$ is called *positive* if the map S_φ is positive, that is, if $T \in \mathcal{B}(L^2(X,\mu))$, $T \geq 0$ implies that $S_\varphi(T) \geq 0$.

Exercise 4.15. *Define an order version of the notion of a matricially norming algebra (see Theorem 2.3) and use it to show the following version of R. R. Smith's theorem (Theorem 2.5):*

If $\varphi \in \mathfrak{S}(X,X)$ and S_φ is positive then S_φ is completely positive.

Exercise 4.16. *Let $\varphi \in \mathfrak{S}(X,X)$. The following are equivalent:*

(i) φ is positive;

(ii) there exists a separable Hilbert space K and an essentially bounded weakly measurable function $a : X \to K$ such that

$$\varphi(x,y) = (a(x), a(y)), \qquad \text{a.e. on } X \times X.$$

Moreover, if (ii) holds true then $\|\varphi\|_{\mathfrak{S}} = \operatorname{esssup}_{x\in X} \|a(x)\|$.

4.5. **Discrete and continuous Schur multipliers.** A particular case of special importance is where X and Y are equipped with the counting measure. In this case, it is convenient to drop the assumption on their σ-finiteness, and this consider arbitrary (and not necessarily countable) sets X and Y.

Exercise 4.17. *Let X and Y be sets. A function $\varphi \in \ell^\infty(X\times Y)$ is a Schur multiplier with respect to the counting measures on X and Y if and only if $(\varphi(x,y)a_{x,y}) \in \mathcal{B}(\ell^2(X), \ell^2(Y))$ whenever $(a_{x,y}) \in \mathcal{B}(\ell^2(X), \ell^2(Y))$.*

We include two characterisation results; for their proofs, we refer the reader to [30].

Theorem 4.18. *Let X (resp. Y) be a locally compact Hausdorff space and μ (resp. ν) be a Radon measure on X (resp. Y) with support equal to X (resp. Y). Let $\varphi : X \times Y \to \mathbb{C}$ be a continuous function. The following are equivalent:*

(i) $\varphi \in \mathfrak{S}_{\mu,\nu}(X,Y)$;

(ii) φ is a Schur multiplier with respect to the counting measures on X and Y.

Theorem 4.19. *Let X (resp. Y) be a locally compact Hausdorff space and μ (resp. ν) be a Radon measure on X (resp. Y) with support equal to X (resp. Y). Let $\varphi : X \times Y \to \mathbb{C}$ be an ω-continuous function. The following are equivalent:*

(i) $\varphi \in \mathfrak{S}_{\mu,\nu}(X,Y)$;

(ii) there exist null sets $M \subseteq X$ and $N \subseteq Y$ such that $\varphi|_{(X\setminus M)\times(Y\setminus N)}$ is a Schur multiplier with respect to the counting measures on $X \setminus M$ and $Y \setminus N$.

We finish this section by recalling a well-known example of a function that is not a Schur multiplier. Let $X = Y = \mathbb{N}$, equipped with counting measure. For a number of questions in Operator Theory, it is important to *truncate* a matrix $A = (a_{i,j})$ of an operator in $\mathcal{B}(\ell^2)$. In other words, given a subset $\kappa \subseteq \mathbb{N} \times \mathbb{N}$, we wish to replace A by the matrix $B = (b_{i,j})$, where $b_{i,j} = a_{i,j}$ if $(i,j) \in \kappa$ and $b_{i,j} = 0$ otherwise. If χ_κ is a Schur multiplier then $B = S_{\chi_\kappa}(A)$ and is hence again a bounded operator on ℓ^2. The question which subsets κ have the property that S_{χ_κ} is a Schur multiplier is still open. (Note that the Schur multipliers that are characteristic functions are precisely the idempotent ones.)

The next theorem is often phrased by saying that triangular truncation is unbounded.

Theorem 4.20. *Let $\kappa = \{(i,j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}$. Then χ_κ is not a Schur multiplier.*

The theorem has a natural measurable version:

Theorem 4.21. *Equip the unit interval $[0,1]$ with Lebesgue measure and let $\kappa = \{(x,y) \in [0,1] \times [0,1] : x \leq y\}$. Then χ_κ is not a Schur multiplier.*

While the statement of Theorem 4.20 requires estimates of matrix norms, its measurable version, Theorem 4.21, can be obtained directly using the results of this section; we suggest its proof as an(other) exercise.

5. Further properties of $M^{\rm cb}A(G)$

5.1. **Embedding into the Schur multipliers.** In this section, we establish a fundamental result due to M. Bożejko-G. Fendler and J. E. Gilbert which establishes an embedding of $M^{\rm cb}A(G)$ into the algebra of Schur multipliers. Let G be a second countable locally compact group equipped with left Haar measure m. We write for short $\mathfrak{S}(G) = \mathfrak{S}_{m,m}(G,G)$.

Recall the usual notation for the map of conjugation by a unitary operator: if U is a unitary operator acting on a Hilbert space H, we let $\mathrm{Ad}_U(T) = UTU^*$, $T \in \mathcal{B}(H)$. Let $\rho : G \to \mathcal{B}(L^2(G))$, $r \to \rho_r$, be the right regular representation of G on $L^2(G)$, that is, the representation given by $(\rho_r f)(s) = \Delta(r)^{1/2} f(sr)$, $s, r \in G$, $f \in L^2(G)$. We recall that

$$\mathrm{VN}(G) = \{\rho_s : s \in G\}'. \tag{15}$$

Given a function $h : G \times G \to \mathbb{C}$ and $r \in G$, let $h_r : G \times G \to \mathbb{C}$ be given by

$$h_r(s,t) = h(sr, tr), \quad s, t \in G.$$

Definition 5.1. *A Schur multiplier $\varphi \in \mathfrak{S}(G)$ will be called* invariant *if $S_\varphi \circ \mathrm{Ad}_{\rho_r} = \mathrm{Ad}_{\rho_r} \circ S_\varphi$ for every $r \in G$.*

We denote by $\mathfrak{S}_{\rm inv}(G)$ the set of all invariant Schur multipliers.

Lemma 5.2. *If $\varphi, \psi \in \mathfrak{S}(G)$ and $S_\varphi(T) = S_\psi(T)$ for all $T \in \mathrm{VN}(G)$ then $\varphi = \psi$.*

Proof. It is well-known (see, *e.g.*, [37, Lemma 3.1]) that the von Neumann algebra generated by $\mathcal{D}_G$ and $\mathrm{VN}(G)$ is $\mathcal{B}(L^2(G))$. (One way to see this is to observe first that $\mathcal{D}_G \cap \mathrm{VN}(G)' = \mathbb{C}I$ and then to take the commutant of this relation.) Note, however, that, if $a \in L^\infty(G)$, $s \in G$ and $a^s \in L^\infty(G)$ is given by $a^s(t) = a(st)$, $t \in G$, then $M_a\lambda_s = \lambda_s M_{a^s}$. It follows that

$$\overline{\mathrm{span}\{M_a\lambda_s : a \in L^\infty(G), s \in G\}}^{\mathrm{w}^*} = \mathcal{B}(L^2(G)).$$

The claim follows from the fact that the maps S_φ and S_ψ are $\mathcal{D}_G$-bimodular and weak* continuous. □

Lemma 5.3. *(i) If* $\varphi \in \mathfrak{S}(G)$ *then* $\varphi_r \in \mathfrak{S}(G)$, $\|\varphi_r\|_{\mathfrak{S}} = \|\varphi\|_{\mathfrak{S}}$ *and* $\mathrm{Ad}_{\rho_r^*} \circ S_\varphi \circ \mathrm{Ad}_{\rho_r} = S_{\varphi_{r^{-1}}}$*;*

(ii) If $h \in T(G)$ *then* $h_r \in T(G)$ *and* $\|h_r\|_t \leq \Delta(r)^{-1}\|h\|_t$*.*

Proof. We only prove (i). For $a \in L^\infty(G)$ let $a_r \in L^\infty(G)$ be given by $a_r(s) = a(sr)$, $s \in G$. A direct verification shows that $\rho_r M_a \rho_r^* = M_{a_r}$. Clearly, if $\varphi = \sum_{k=1}^\infty a_k \otimes b_k$ as in Theorem 4.13 then $\varphi_r = \sum_{k=1}^\infty (a_k)_r \otimes (b_k)_r$. Now, if $T \in \mathcal{B}(L^2(G))$ then

$$\begin{aligned}\mathrm{Ad}\,_{\rho_r^*} \circ S_\varphi \circ \mathrm{Ad}\,_{\rho_r}(T) &= \sum_{k=1}^\infty (\rho_r^* M_{b_k}\rho_r) T (\rho_r^* M_{a_k}\rho_r) \\ &= \sum_{k=1}^\infty M_{(b_k)_{r^{-1}}} T M_{(a_k)_{r^{-1}}} = S_{\varphi_{r^{-1}}}(T).\end{aligned}$$

□

Lemma 5.4. *A Schur multiplier* φ *is invariant if and only if* S_φ *leaves* $\mathrm{VN}(G)$ *invariant.*

Proof. If φ is an invariant Schur multiplier and $T \in \mathrm{VN}(G)$ then, by (15),

$$S_\varphi(T) = S_\varphi(\rho_r T \rho_r^*) = \rho_r S_\varphi(T)\rho_r^*, \quad r \in G.$$

Thus, $S_\varphi(T)$ commutes with ρ_r, $r \in G$; by (15) again, it belongs to $\mathrm{VN}(G)$.

Conversely, assume that S_φ leaves $\mathrm{VN}(G)$ invariant. If $T \in \mathrm{VN}(G)$ then $S_\varphi(T) \in \mathrm{VN}(G)$ and hence

$$S_\varphi(\rho_r T \rho_r^*) = S_\varphi(T) = \rho_r S_\varphi(T)\rho_r^*, \quad r \in G.$$

Thus,

$$\rho_r^* S_\varphi(\rho_r T \rho_r^*)\rho_r = S_\varphi(T), \quad T \in \mathrm{VN}(G).$$

By Lemma 5.3,

$$S_{\varphi_{r^{-1}}}(T) = S_\varphi(T), \quad T \in \mathrm{VN}(G)$$

and now Lemma 5.2 implies that

$$S_{\varphi_{r^{-1}}}(T) = S_\varphi(T), \quad \text{for all} \quad T \in \mathcal{B}(L^2(G)).$$

Using Lemma 5.3 again, we obtain that

$$S_\varphi(\rho_r T \rho_r^*) = \rho_r S_\varphi(T)\rho_r^*, \quad \text{for all} \quad T \in \mathcal{B}(L^2(G)),$$

and hence φ is an invariant Schur multiplier. □

The proof of Lemma 5.4 implies the following.

Corollary 5.5. *The following are equivalent, for an element* $\varphi \in \mathfrak{S}(G)$*:*
(i) $\varphi \in \mathfrak{S}_{\rm inv}(G)$*;*
(ii) $\varphi \sim \varphi_r$*, for every* $r \in G$*.*

Given a function $u : G \to \mathbb{C}$, let $N(u) : G \times G \to \mathbb{C}$ be the function defined by

$$N(u)(s,t) = u(ts^{-1}), \quad s,t \in G.$$

It is clear that if u is measurable (resp. continuous) then $N(u)$ is measurable (resp. continuous).

The next theorem is one of the main results in this section. For its proof, we follow [28].

Theorem 5.6. *The map* N *is a surjective isometry from* $M^{\rm cb}A(G)$ *onto* $\mathfrak{S}_{\rm inv}(G)$*.*

Proof. Let $u \in M^{\rm cb}A(G)$. Recall from Section 3 that we denote by S_u the dual, acting on ${\rm VN}(G)$, of the multiplication map m_u. Let $\Phi : C_r^*(G) \to C_r^*(G)$ be the restriction of Φ to the reduced C*-algebra of G. By Theorem 2.2, there exists a Hilbert space K, a non-degenerate *-representation $\pi : C_r^*(G) \to \mathcal{B}(K)$ and bounded operators $V, W : L^2(G) \to K$ such that

$$\Phi(a) = W^*\pi(a)V, \ a \in C_r^*(G), \ \text{ and } \ \|\Phi\|_{\rm cb} = \|V\|\|W\|.$$

Set $\tilde{\pi}$ be the *-representation of $L^1(G)$ given by $\tilde{\pi}(f) = \pi(\lambda(f))$. The representation $\tilde{\pi}$ arises from a unitary representation of G on K, which will be denoted again by $\tilde{\pi}$; thus, $\tilde{\pi}(f) = \int_G f(s)\tilde{\pi}(\lambda_s)ds$, $f \in L^1(G)$.

We have

$$\Phi(\lambda(f)) = W\pi(\lambda(f))V, \quad f \in L^1(G). \tag{16}$$

Fix $s \in G$ and, for each compact neighbourhood V of s, let $f_V \in L^1(G)$ be a function taking non-negative values such that $\|f_V\|_1 = 1$. Then $\lambda(f_V) \to_V \lambda_s$ and $\tilde{\pi}(f_V) \to \tilde{\pi}(s)$ in the weak* topology (we leave these facts as an exercise). Since Φ is weak* continuous and $S_u(\lambda_s) = u(s)\lambda_s$, (16) shows that

$$u(s)\lambda_s = \Phi(\lambda_s) = W\tilde{\pi}(s)V, \quad s \in G. \tag{17}$$

Let $\zeta \in L^2(G)$ be a unit vector and, for each $s \in G$, set

$$\xi(s) = \tilde{\pi}(s^{-1})V\lambda_s\zeta, \quad \eta(s) = \tilde{\pi}(s^{-1})W\lambda_s\zeta.$$

Thus, $\xi, \eta : G \to K$ are continuous (and hence weakly measurable) vector-valued functions with

$$\sup{}_{s\in G}\|\xi(s)\| = \sup{}_{s\in G}\|\tilde{\pi}(s^{-1})V\lambda_s\zeta\| \le \|V\|$$

and, similarly,

$$\sup{}_{s\in G}\|\eta(s)\| \le \|W\|.$$

By (17),

$$\begin{aligned}(\xi(s),\eta(t)) &= (\tilde{\pi}(s^{-1})V\lambda_s\zeta, \tilde{\pi}(t^{-1})W\lambda_t\zeta)\\ &= (W^*\tilde{\pi}(ts^{-1})V\lambda_s\zeta, \lambda_t\zeta)\\ &= u(ts^{-1})(\lambda_{ts^{-1}}\lambda_s\zeta, \lambda_t\zeta) = u(ts^{-1}).\end{aligned}$$

By Theorem 4.13, $N(u) \in \mathfrak{S}(G)$ and

$$\|N(u)\|_{\mathfrak{S}} \leq \|V\|\|W\| = \|\Phi\|_{\mathrm{cb}} = \|u\|_{\mathrm{cbm}}. \tag{18}$$

By Corollary 5.5, $N(u) \in \mathfrak{S}_{\mathrm{inv}}(G)$.

Note next that $S_{N(u)}$ extends S_u. Indeed, let $f \in C_c(G)$. If $g, h \in L^2(G)$ and $L \subseteq G$ is compact, then

$$\begin{aligned}(M_{\chi_L}\lambda(f)M_{\chi_L}g, h) &= \int_{G\times G} f(s)\chi_L(s^{-1}t)g(s^{-1}t)\chi_L(t)\overline{h(t)}dsdt\\ &= \int_{LL^{-1}\times L} f(s)g(s^{-1}t)\overline{h(t)}dsdt\\ &= \int_{LL^{-1}L\times L} \Delta(r^{-1})f(tr^{-1})g(r)\overline{h(t)}drdt\\ &= \int_{G\times G} \chi_{LL^{-1}L\times L}(r,t)\Delta(r^{-1})f(tr^{-1})g(r)\overline{h(t)}drdt.\end{aligned}$$

The function $k : G\times G \to \mathbb{C}$ given by $k(t,r) = \chi_{LL^{-1}L\times L}(r,t)\Delta(r^{-1})f(tr^{-1})$ belongs to $L^2(G \times G)$, and $M_{\chi_L}\lambda(f)M_{\chi_L} = T_k$. It follows that

$$S_{N(u)}(M_{\chi_L}\lambda(f)M_{\chi_L}) = T_{\widehat{N(u)}k} = M_{\chi_L}\lambda(uf)M_{\chi_L}.$$

Since this holds for all compact sets L, we conclude that $S_{N(u)}(\lambda(f)) = \lambda(uf)$ for all $f \in C_c(G)$. Thus, $S_{N(u)}(\lambda(f)) = u \cdot \lambda(f)$ for all $f \in C_c(G)$; since the set $\{\lambda(f) : f \in C_c(G)\}$ is dense in $\mathrm{VN}(G)$ in the weak* topology, we obtain, using Proposition 3.10 and the weak* continuity of $S_{N(u)}$ and S_u, that

$$S_{N(u)}(T) = u \cdot T = S_u(T), \qquad T \in \mathrm{VN}(G). \tag{19}$$

Thus, $S_{N(u)}$ extends S_u.

Suppose that $\varphi \in \mathfrak{S}_{\mathrm{inv}}(G)$. By Lemma 5.4, S_φ leaves $\mathrm{VN}(G)$ invariant; let $\Phi = S_\varphi|_{\mathrm{VN}(G)}$. If $u \in A(G)$ and $T \in \mathrm{VN}(G)$ then, using (19), we have

$$\Phi(u \cdot T) = S_\varphi(S_{N(u)}(T)) = S_{N(u)}(S_\varphi(T)) = u \cdot \Phi(T).$$

By Exercise 3.3, there exists an element $v \in MA(G)$ such that $S_v = \Phi$. Since Φ is a completely bounded map, $v \in M^{\mathrm{cb}}A(G)$. Now,

$$S_{N(v)}(T) = S_v(T) = S_\varphi(T), \quad T \in \mathrm{VN}(G).$$

By Lemma 5.2, $\varphi = N(v)$ and thus the map N is onto $\mathfrak{S}_{\mathrm{inv}}(G)$. Moreover,

$$\|v\|_{\mathrm{cbm}} = \|S_v\|_{\mathrm{cb}} \leq \|S_{N(v)}\|_{\mathrm{cb}} = \|N(v)\|_{\mathfrak{S}},$$

where the inequality follows from the fact that $S_{N(v)}$ extends S_v. Combined with (18), this shows that N is an isometry from $M^{\mathrm{cb}}A(G)$ onto $\mathfrak{S}_{\mathrm{inv}}(G)$. $\square$

The proof of Theorem 5.6 yields the following fact.

Remark 5.7. *If $u \in M^{\rm cb}A(G)$ then there exist bounded* and continuous *functions $\xi, \eta : G \to L^2(G)$ such that $N(u)(s,t) = (\xi(s), \eta(t))$, $s,t \in G$.*

Remark 5.8. *A continuous function $u : G \to \mathbb{C}$ belongs to $M^{\rm cb}A(G)$ if and only if the function $(s,t) \to u(s^{-1}t)$ is a Schur multiplier.*

Proof. If $u(ts^{-1}) = (\xi(s), \eta(t))$, $s,t \in G$, for some weakly measurable Hilbert space valued functions ξ and η, then $u(s^{-1}t) = (\xi(t^{-1}), \eta(s^{-1}))$, $s,t \in G$, and the functions $s \to \eta(s^{-1})$ and $t \to \xi(t^{-1})$ are weakly measurable. □

Corollary 5.9. *Let $u \in L^\infty(G)$. The following are equivalent:*
(i) $u \in^m M^{\rm cb}A(G)$;
(ii) $N(u) \in \mathfrak{S}(G)$.
Moreover, if (i) holds then $\|u\|_{\rm cbm} = \|N(u)\|_{\mathfrak{S}}$.

Proof. (i)⇒(ii) Let $v \in M^{\rm cb}A(G)$ be a function such that $u \sim v$. Then $N(u) \sim N(v)$; by Theorem 5.6, $N(u) \in \mathfrak{S}_{\rm inv}(G)$.

(ii)⇒(i) Suppose $N(u) \in \mathfrak{S}(G)$. By Corollary 5.5, $N(u)$ is an invariant Schur multiplier. By Theorem 5.6, there exists $v \in M^{\rm cb}A(G)$ such that $N(v) \sim N(u)$. It follows that $u \sim v$. □

Exercise 5.10 ([48])**.** *Show that N is in fact a complete isometry.*

We note two useful consequences of Theorem 5.6.

Corollary 5.11. *Let G be a locally compact group and H be a closed subgroup of G. If $u \in M^{\rm cb}A(G)$ then the restriction $u|_H$ of u to H belongs to $M^{\rm cb}A(H)$, and $\|u|_H\|_{\rm cbm} \le \|u\|_{\rm cbm}$.*

Proof. Immediate from Theorem 5.6. □

Corollary 5.12. *Let G and H be locally compact groups, $u \in M^{\rm cb}A(G)$ and $v \in M^{\rm cb}A(H)$. Then the function $u \otimes v$ (given by $u \otimes v(s,x) = u(s)v(x)$) belongs to $M^{\rm cb}A(G \times H)$ and $\|u \otimes v\|_{\rm cbm} = \|u\|_{\rm cbm}\|v\|_{\rm cbm}$.*

Proof. Using Theorem 5.6, write

$$u(ts^{-1}) = (p(s), q(t)),\ \ s,t \in G,\ \text{ and } v(yx^{-1}) = (p'(x), q'(y)),\ \ x,y \in H,$$

where $p, q : G \to \ell^2$ and $p', q' : H \to \ell^2$ are bounded functions with

$$\|u\|_{\rm cbm} = \|p\|_\infty \|q\|_\infty \text{ and } \|v\|_{\rm cbm} = \|p'\|_\infty \|q'\|_\infty.$$

Let $f, g : G \times H \to \ell^2 \otimes \ell^2$ be given by $f(s,x) = p(s) \otimes p'(x)$ and $g(s,x) = q(s) \otimes q'(x)$. Then $\|f\|_\infty = \|p\|_\infty \|p'\|_\infty$, $\|g\|_\infty = \|q\|_\infty \|q'\|_\infty$ and

$$(u \otimes v)((t,y)^{-1}(s,x)) = u(ts^{-1})v(yx^{-1}) = (f(s,x), g(t,y))_{\ell^2 \otimes \ell^2}.$$

By Theorem 5.6, $u \otimes v \in M^{\rm cb}A(G \times H)$ and $\|u \otimes v\|_{\rm cbm} \le \|u\|_{\rm cbm}\|v\|_{\rm cbm}$.

It follows that the operator $S_u \otimes S_v$, defined on the algebraic tensor product $\mathrm{VN}(G) \otimes \mathrm{VN}(H)$, admits an extension to $\mathrm{VN}(G \times H)$ which coincides with $S_{u \otimes v}$. It now follows that $\|S_{u \otimes v}\|_{\rm cb} \ge \|S_u\|_{\rm cb}\|S_v\|_{\rm cb}$ and we conclude that $\|u \otimes v\|_{\rm cbm} = \|u\|_{\rm cbm}\|v\|_{\rm cbm}$. □

If $u \in M^{\rm cb}A(G)$ then $N(u)$ is continuous and hence ω-continuous. It is perhaps surprising that the latter condition alone suffices to ensure the continuity of the function u; in fact, if $u : G \to \mathbb{C}$ is a measurable function then it can be shown [46] that $N(u)$ is ω-continuous if and only if u is continuous.

5.2. **The case of compact groups.** Let the group G be compact. In this case, the Haar measure m is a probability measure and $1 \in A(G)$. Thus, if $u \in M^{\rm cb}A(G)$ then $u = m_u(1) \in A(G)$; so, $M^{\rm cb}A(G) = A(G)$. Moreover, $\|u\|_{A(G)} \leq \|u\|_{\rm cbm}$.

Similarly, the constant function on $G \times G$ taking value 1 belongs to $T(G)$ (it coincides with the elementary tensor $1 \otimes 1$). Thus, if $\varphi \in \mathfrak{S}(G)$ then $\varphi = m_\varphi(1 \otimes 1) \in T(G)$; moreover,

$$\|\varphi\|_\wedge \leq \|\varphi\|_{\mathfrak{S}}. \tag{20}$$

We thus have that N maps $A(G)$ into $T(G)$. In the reverse direction, we have the contraction $P : T(G) \to A(G)$ defined by

$$P(f \otimes g)(s) = \langle \lambda_s, f \otimes g\rangle, \quad f, g \in L^2(G), s \in G.$$

Proposition 5.13. *We have that $P \circ N = {\rm id}_{A(G)}$. Thus, N is an isometry when considered as a map from $A(G)$ into $T(G)$.*

Proof. Let $u \in A(G)$. By Lemma 4.7,

$$P(N(u))(t) = \int_G N(u)(t^{-1}s, s)ds = \int_G u(t)ds = u(t), \quad t \in G.$$

Moreover,

$$\begin{aligned} \|u\|_{A(G)} &= \|P(N(u))\|_{A(G)} \leq \|N(u)\|_\wedge \leq \|N(u)\|_{\mathfrak{S}} \\ &= \|u\|_{\rm cbm} \leq \|u\|_{B(G)} = \|u\|_{A(G)} \end{aligned}$$

and hence we have equalities throughout. □

5.3. **Coefficients of representations.** The following corollary gives a supply of examples of Herz-Schur multipliers.

Corollary 5.14. *Let G be a locally compact group and $\pi : G \to \mathcal{B}(H)$ be a strongly continuous uniformly bounded (not necessarily unitary) representation, with $\sup_{s\in G} \|\pi(s)\| = C < \infty$. Let $\xi, \eta \in H$ and $u(s) = (\pi(s)\xi, \eta)$, $s \in G$. Then $u \in M^{\rm cb}A(G)$ and $\|u\|_{\rm cbm} \leq C^2\|\xi\|\|\eta\|$.*

Proof. We have

$$N(u)(s, t) = (\pi(ts^{-1})\xi, \eta) = (\pi(s^{-1})\xi, \pi(t^{-1})\eta), \quad s, t \in G,$$

$$\sup_{s\in G} \|\pi(s^{-1})\xi\| \leq C\|\xi\| \ \text{ and } \ \sup_{t\in G} \|\pi(t^{-1})\eta\| \leq C\|\eta\|.$$

The claim now follows from Corollary 5.9 and Theorem 4.13. □

A naturally arising question is whether the class of Herz-Schur multipliers exhibited in Corollary 5.14 contains functions outside of the Fourier-Stieltjes algebra $B(G)$. To this end, we note the following result, established in [5]:

Theorem 5.15. *Let G be a locally compact group and $\pi : G \to \mathcal{B}(H)$ be a strongly continuous uniformly bounded cyclic representation. The following are equivalent:*
(i) All coefficients of π belong to $B(G)$;
(ii) π is similar to a unitary representation of G.

The above result implies that, for a number of groups, $M^{\text{cb}}A(G) \neq B(G)$; examples of such groups are $SL(2,\mathbb{R})$, $SL(n,\mathbb{C})$ $(n \geq 2)$, $O(n,\mathbb{C})$ $(n \geq 5)$. It was shown in [5] that, moreover, if H is a closed normal subgroup of G for which G/H is isomorphic to any of the groups listed above then $M^{\text{cb}}A(G) \neq B(G)$.

For $c \geq 1$, let us denote by $B_c(G)$ the set of all coefficients of strongly continuous representations $\pi : G \to \mathcal{B}(H)$ such that $\sup_{s\in G} \|\pi(s)\| \leq c$. Since a representation π is unitary if and only if $\sup_{s\in G} \|\pi(s)\| = 1$, we have that $B_1(G) = B(G)$. By Corollary 5.14,

$$B(G) \subseteq \cap_{c>1} B_c(G) \subseteq \cup_{c>1} B_c(G) \subseteq M^{\text{cb}}A(G) \subseteq MA(G). \tag{21}$$

The natural question that arises is whether the above inclusions are strict. To this end, we have the following result (recall that $\mathbb{F}_n$ denotes the free group on n generators).

Theorem 5.16. *If $G = \mathbb{F}_n$ with $n > 1$ then the inclusions in (21) are proper.*

The fact that the first inclusion is strict, in the case $G = \mathbb{F}_n$, follows from [3, Corollary 2.2]. The fact that the third inclusion is proper was proved by U. Haagreup [18] and the fact that the last inclusion is proper can be found in [2] and [23], among others (see also Theorem 6.13).

Although there are Herz-Schur multipliers that do not arise as coefficients of uniformly bounded representations (even for discrete groups), we have the results below in the positive direction. We will need the following fact [3].

Exercise 5.17. *Let K be a subgroup of a discrete group G. For $u \in M^{\text{cb}}A(K)$, let v be the function on G that coincides with u on K and is zero on $G \setminus K$. Then $v \in M^{\text{cb}}A(G)$ and $\|v\|_{\text{cbm}} \leq \|u\|_{\text{cbm}}$.*

Hint. Write $G = \cup_{i\in I} x_i K$ as a disjoint union of left cosets of K. If u admits a representation in the form $u(yx^{-1}) = (a(x), b(y))$, where $a, b : K \to H$ and H is a Hilbert space, work with the Hilbert space $\mathcal{H} = \oplus_{i\in I} H$, direct sum of $|I|$ copies of H.

The next theorem was established in [3].

Theorem 5.18. *Let G be a countable discrete group. The following are equivalent, for a function $u : G \to \mathbb{C}$:*

(i) $u \in M^{\rm cb}A(G)$;

(ii) for every $\epsilon > 0$, there exists a (not necessarily uniformly bounded) representation $\pi : G \to \mathcal{B}(H)$ and vectors $\xi, \eta \in H$ such that

$$\sup_{s\in G} \|\pi(s)\xi\| \leq 2(1+\epsilon)\|u\|_{\rm cbm}, \quad \sup_{t\in G} \|\pi(t)\eta\| \leq 2(1+\epsilon)\|u\|_{\rm cbm}$$

and

$$u(s^{-1}t) = (\pi(s)\xi, \pi(t)\eta), \quad s,t \in G. \tag{22}$$

Proof. (ii)⇒(i) follows from Theorem 5.6.

(i)⇒(ii) It suffices to show that if u is hermitian, that is, $u(x) = u^*(x) \stackrel{def}{=} \overline{u(x^{-1})}$ ($x \in G$) then there exists a representation π on a Hilbert space H and vectors $\xi, \eta \in H$ such that

$$\sup_{s\in G} \|\pi(s)\xi\| \leq (1+\epsilon)\|u\|_{\rm cbm}, \quad \sup_{t\in G} \|\pi(t)\eta\| \leq (1+\epsilon)\|u\|_{\rm cbm}$$

and (24) holds. Indeed, every multiplier u can be written as a sum $\frac{1}{2}(u + u^*) + i\frac{1}{2i}(u - u^*)$ of two hermitian ones. A standard direct sum argument then implies the statement for a general u.

So assume that u is hermitian and that $\|u\|_{\rm cbm} = \frac{1}{1+\epsilon}$ for some $\epsilon > 0$. Let H be a Hilbert space and $a, b : G \to H$ be functions with $\sup_{x\in G} \|a(x)\| \leq 1$, $\sup_{x\in G} \|b(x)\| \leq 1$ and

$$u(x^{-1}y) = (a(x), b(y)), \quad x, y \in G.$$

For $x, y \in G$, set

$$a_1(x,y) = \frac{1}{4}(a(x)+b(x), a(y)+b(y)), \quad a_2(x,y) = \frac{1}{4}(a(x)-b(x), a(y)-b(y)).$$

Clearly, a_1 and a_2 are positive definite functions on $G \times G$. Moreover, since $u(x^{-1}y) = \overline{u(y^{-1}x)}$, we have that $(a(x), b(y)) = (b(x), a(y))$, $x, y \in G$. It follows that

$$u(x^{-1}y) = a_1(x,y) - a_2(x,y), \quad x, y \in G.$$

Moreover,

$$\begin{aligned} a_1(x,x) &= \frac{1}{4}\|a(x)+b(x)\|^2 = \frac{1}{4}(\|a(x)\|^2 + \|b(x)\|^2 + 2Re(a(x), b(x))) \\ &\leq \frac{1}{4}(2 + 2Re(u(e))); \end{aligned}$$

similarly,

$$a_2(x,x) \leq \frac{1}{4}(2 - 2Re(u(e)))$$

and thus

$$\sup_{x\in G} a_1(x,x) + \sup_{x\in G} a_2(x,x) \leq 1 = (1+\epsilon)\|u\|_{\rm cbm}.$$

By [26], the group G can be embedded in a group generated by two elements, say a and b. By Exercise 5.17, we may assume that the group G itself is generated by a and b.

Let $E = \{a, b, a^{-1}, b^{-1}\}$, $\mu_0 = \delta_e$ be the mass point measure at the neutral element e and

$$\mu = |E|^{-1} \sum_{x \in E} \delta_x.$$

If $c : G \times G \to \mathbb{C}$ is a bounded function, and $\nu = \sum_{i=1}^{k} \lambda_i \delta_{z_i}$, where $\lambda_i \in \mathbb{C}$ and $z_i \in G$, $i = 1, \dots, n$, let

$$\nu \circ c(x, y) = \sum_{i=1}^{k} \lambda_k c(z_i x, z_i y), \quad x, y \in G.$$

Thus,

$$\mu \circ c(x, y) = |E|^{-1} \sum_{z \in E} c(zx, zy), \quad x, y \in G.$$

It is easy to see that if c is positive definite then so is $\mu \circ c$. Let μ^n be the nth convolution power of μ and set

$$S_{x,y,i}(n) = (\mu^n \circ a_i)(x, y), \quad x, y \in G, \ i = 1, 2.$$

Let M be a Banach limit on ℓ^∞ and set

$$\tilde{a}_i(x, y) = M((S_{x,y,i}(n)_{n \in \mathbb{N}}), \quad i = 1, 2.$$

We have that

(1) $\tilde{a}_1$ and $\tilde{a}_2$ are positive definite bounded functions on $G \times G$,
(2) $\sup_{x \in G} \tilde{a}_i(x, x) \leq \sup_{x,y \in G} a_i(x, y) \leq \sup_{x \in G} a_i(x, x) \leq 1$, $i = 1, 2$,
(3) $\mu^n \circ \tilde{a}_i = \tilde{a}_i$, $i = 1, 2$, $n \in \mathbb{N}$,
(4) $u(x^{-1}y) = \tilde{a}_1(x, y) - \tilde{a}_2(x, y)$, $x, y \in G$.

Indeed, (1) and (2) are straightforward. Identity (3) follows from the shift invariance of the functional M. Finally, (4) follows from the fact that, for every $z \in G$ we have

$$a_1(zx, zy) - a_2(zx, zy) = u((zx)^{-1}(zy)) = u(y^{-1}x);$$

thus

$$u(x^{-1}y) = \mu \circ a_1 - \mu \circ a_2$$

and hence

$$u(xy^{-1}) = S_{x,y,1}(n) - S_{x,y,2}(n), \quad n \in \mathbb{N}.$$

Let $K(G)$ be the vector space of all finitely supported functions on G (that is, $K(G) = C_c(G)$). For $f, g \in K(G)$ and a positive definite function $c : G \times G \to \mathbb{C}$, write

$$c(f, g) = \sum_{x,y \in G} c(x, y) f(x) g(y).$$

Also, for $x \in G$ and $f \in K(G)$, let $xf = \lambda_x(f)$; thus, $xf(y) = f(x^{-1}y)$, $y \in G$. Relation (3) implies

$$|E|^{-n} \sum_{z_1,\dots,z_n \in E} \tilde{a}_i(z_1 \dots z_n f, z_1 \dots z_n g) = \tilde{a}_i(f,g), \quad f,g \in K(G). \tag{23}$$

Indeed, the left hand side of (23) is easily seen to be equal to $(\mu^n \circ \tilde{a}_i)(f,g)$.

Let H_i be the vector space $K(G)$ equipped with the inner product

$$(f,g)_i = \tilde{a}_i(f,\bar{g}), \quad f,g \in K(G), \quad i = 1,2.$$

(Since $\tilde{a}_i$ is positive definite, we have that the above formula indeed defines an inner product.) If $x \in E^n$ then, by (23),

$$\begin{aligned} 0 &\leq \tilde{a}_i(xf, x\bar{f}) \leq \sum_{z \in E^n} \tilde{a}_i(zf, z\bar{f}) \leq \sum_{z_1,\dots,z_n \in E} \tilde{a}_i(z_1 \dots z_n f, z_1 \dots z_n \bar{f}) \\ &= |E|^n \tilde{a}_i(f,\bar{f}). \end{aligned}$$

Denote by L the left regular representation of G on $K(G)$; thus, $L_x(f) = xf$, $x \in G$. Setting $N_i = \{f \in H_i : (f,f)_i = 0\}$, the above inequalities now imply that N_i is invariant under L_x for all $x \in G$ (here we use the fact that $G = \cup_{n \geq 0} E^n$). Set $\tilde{H}_i = H_i/N_i$, $i = 1,2$ and $\tilde{H} = \tilde{H}_1 \oplus \tilde{H}_2$. Let $\pi_i : G \to \mathcal{B}(\tilde{H}_i)$ be the representation given by

$$\pi_i(x)(f + N_i) = L_x(f) + N_i, \quad x \in G, \ i = 1,2.$$

Set $\pi = \pi_1 \oplus \pi_2$, $\tilde{\delta}_i = \delta_e + N_i \in \tilde{H}_i$, $i = 1,2$,

$$\xi = \tilde{\delta}_1 \oplus \tilde{\delta}_2, \quad \eta = \tilde{\delta}_1 \oplus (-\tilde{\delta}_2).$$

We have

$$\begin{aligned} (\pi(x)\xi, \pi(y)\eta) &= ((\pi_1(x) \oplus \pi_2(x))(\tilde{\delta}_1 \oplus \tilde{\delta}_2), (\pi_1(y) \oplus \pi_2(y))(\tilde{\delta}_1 \oplus (-\tilde{\delta}_2)) \\ &= (\delta_x, \delta_y)_1 - (\delta_x, \delta_y)_2 = \tilde{a}_1(x,y) - \tilde{a}_2(x,y) = u(x^{-1}y). \end{aligned}$$

Moreover, for $x \in G$ we have

$$\|\pi(x)\xi\|^2 = \|\pi_1(x)\tilde{\delta}_e\|^2 + \|\pi_2(x)\tilde{\delta}_e\|^2 = \tilde{a}_1(x,x) + \tilde{a}_2(x,x) \leq (1+\epsilon)\|f\|_{\text{cbm}}.$$

Similarly, one shows that $\|\pi(x)\xi\|^2 \leq (1+\epsilon)\|f\|_{\text{cbm}}$ for every $x \in G$, and the proof is complete. □

A further extension of Theorem 5.18 was established by T. Steenstrup in [50].

Theorem 5.19. *Let G be a second countable locally compact group. The following are equivalent, for a function $u : G \to \mathbb{C}$:*

(i) $u \in M^{\text{cb}}A(G)$;

(ii) there exists a (not necessarily uniformly bounded) representation $\pi : G \to \mathcal{B}(H)$ and vectors $\xi, \eta \in H$ such that

$$\sup_{s \in G} \|\pi(s)\xi\| = \sqrt{\|u\|_{\text{cbm}}}, \quad \sup_{t \in G} \|\pi(t)\eta\| = \sqrt{\|u\|_{\text{cbm}}}$$

and

$$u(t^{-1}s) = (\pi(s)\xi, \pi(t)\eta), \quad s,t \in G. \tag{24}$$

5.4. **The canonical predual of $M^{\rm cb}A(G)$.** For $f \in L^1(G)$, we define

$$\|f\|_{\rm pred} = \sup\left\{\left|\int_G f(x)u(x)dx\right| : u \in M^{\rm cb}A(G), \|u\|_{\rm cbm} \leq 1\right\}.$$

It is easy to observe that $\|\cdot\|_{\rm pred}$ is a norm on $L^1(G)$; let $\mathcal{Q}(G)$ be the completion of $L^1(G)$ with respect to $\|\cdot\|_{\rm pred}$. By (4), $\|u\|_\infty \leq \|u\|_{\rm cbm}$ for every $u \in M^{\rm cb}A(G)$, and hence

$$\|f\|_{\rm pred} \leq \|f\|_1, \quad f \in L^1(G). \tag{25}$$

Lemma 5.20. *If $(u_i)_i \subseteq M^{\rm cb}A(G)$ is a net with $\|u_i\|_{\rm cbm} \leq 1$ for all i, $u \in L^\infty(G)$ and $u_i \to u$ in the weak* topology of $L^\infty(G)$, then u is almost everywhere equal to an element from $M^{\rm cb}A(G)$ and $\|u\|_{\rm cbm} \leq 1$.*

Proof. We have

$$(\lambda(u_i f)\xi, \eta) = \int_G u_i(s)f(s)(\xi * \tilde{\eta})(s)ds, \quad \xi, \eta \in L^2(G), f \in L^1(G),$$

where $\tilde{\eta}(x) = \overline{\eta(x^{-1})}$, $x \in G$. Note that, since $\xi * \tilde{\eta}$ is a bounded function, $f(\xi * \tilde{\eta})$ belongs to $L^1(G)$; thus, the assumption implies that

$$\begin{aligned} |(\lambda(uf)\xi, \eta)| &= \left|\int_G u(s)f(s)(\xi * \tilde{\eta})(s)ds\right| \\ &\leq \limsup |(\lambda(u_i f)\xi, \eta)| \leq \|\lambda(f)\|\|\xi\|_2\|\eta\|_2 \end{aligned}$$

and the claim follows from Remark 3.7. □

Proposition 5.21. *The Banach space dual $\mathcal{Q}(G)^*$ of $\mathcal{Q}(G)$ is isometrically isomorphic to $M^{\rm cb}A(G)$.*

Proof. Let $\omega \in \mathcal{Q}(G)^*$ have norm one. Then

$$|\omega(f)| \leq \|f\|_{\rm pred} \leq \|f\|_1, \quad f \in L^1(G).$$

Thus there exists $u \in L^\infty(G)$ such that

$$\omega(f) = \int_G uf dm, \quad f \in L^1(G). \tag{26}$$

Since $\|\omega\| = 1$, we have

$$\|f\|_{\rm pred} \leq 1 \implies \left|\int_G uf dm\right| \leq 1.$$

By the Hahn-Banach Theorem, u is in the weak* closure in $L^\infty(G)$ of the unit ball of $M^{\rm cb}A(G)$. By Lemma 5.20, u is equivalent to an element of the unit ball of $M^{\rm cb}A(G)$.

Conversely, if $u \in M^{\rm cb}A(G)$ has norm one, then the definition of $\|\cdot\|_{\rm pred}$ implies that the functional on $\mathcal{Q}(G)$ defined through (26) has norm one. □

The space $\mathcal{Q}(G)$ has two more useful descriptions, which we include here. The first one is due to U. Haagerup and J. Kraus [22]. Let H be a fixed infinite dimensional separable Hilbert space. For an element $a \in C_r^*(G) \otimes_{\min} \mathcal{K}(H)$ and an element $\varphi \in (\mathrm{VN}(G)\bar{\otimes}\mathcal{B}(H))_*$, let $\omega_{a,\varphi} : M^{\mathrm{cb}}A(G) \to \mathbb{C}$ be given by

$$\omega_{a,\varphi}(u) = \langle (S_u \otimes \mathrm{id})(a), \varphi \rangle, \qquad u \in M^{\mathrm{cb}}A(G)\,.$$

Clearly,

$$|\omega_{a,\varphi}(u)| \leq \|S_u \otimes \mathrm{id}\,\|\|a\|\|\varphi\| = \|u\|_{\mathrm{cbm}}\|a\|\|\varphi\|.$$

Thus, $\omega_{a,\varphi}$ is a bounded functional on $M^{\mathrm{cb}}A(G)$ and

$$\|\omega_{a,\varphi}\| \leq \|a\|\|\varphi\|.$$

In fact, we have the following result:

Theorem 5.22. *The elements of $\mathcal{Q}(G)$ are precisely the functionals on $M^{\mathrm{cb}}A(G)$ of the form $\omega_{a,\varphi}$ for some $a \in C_r^*(G) \otimes_{\min} \mathcal{K}(H)$ and some $\varphi \in (\mathrm{VN}(G)\bar{\otimes}\mathcal{B}(H))_*$.*

We will not give here the proof of this theorem, but we suggest as an exercise the easier implication, namely, that all $\omega_{a,\varphi}$ belong to $\mathcal{Q}(G)$.

We close this section with yet a third view on the predual of $M^{\mathrm{cb}}A(G)$, described in [48]. We first recall that the space $\mathfrak{S}(G)$ of all Schur multipliers on $G \times G$ can be identified in a natural fashion with the weak* Haagerup tensor product $L^\infty(G) \otimes_{\mathrm{w}^*\mathrm{h}} L^\infty(G)$. On the other hand, the latter space is the dual of the Haagerup tensor product $L^1(G) \otimes_{\mathrm{h}} L^1(G)$, where $L^1(G)$ is equipped with the operator space structure arising from the identification $L^1(G)^* = L^\infty(G)$. The duality between $L^1(G) \otimes_{\mathrm{h}} L^1(G)$ and $\mathfrak{S}(G)$ is given as follows:

$$\langle \varphi, f \otimes g \rangle = \int_{G\times G} \varphi(s,t) f(s) g(t) ds dt, \quad \varphi \in \mathfrak{S}(G), f, g \in L^1(G).$$

Let $m : L^1(G) \otimes L^1(G) \to L^1(G)$ be the linear map given on elementary tensors by

$$m(f \otimes g) = f * g, \qquad f, g \in L^1(G);$$

thus,

$$m(f \otimes g)(t) = \int_G f(s) g(s^{-1}t) ds, \quad t \in G.$$

Set $\mathcal{K}_0 = \ker m$ and $\mathcal{K} = \overline{\mathcal{K}_0}$, where the closure is taken with respect to the Haagerup norm in $L^1(G) \otimes_{\mathrm{h}} L^1(G)$.

For $\varphi \in \mathfrak{S}(G)$ let $\tilde{\varphi}$ be the function given by $\tilde{\varphi}(s,t) = \varphi(t^{-1}, s)$. It is easy to see that the mapping $\varphi \to \tilde{\varphi}$ is a weak* continuous surjective isometry on $\mathfrak{S}(G)$. Let $\tilde{N}$ be the map sending a function $u : G \to \mathbb{C}$ to the function $\widetilde{N(u)}$. Thus, $\tilde{N}(u)(s,t) = u(st)$, $s, t \in G$. Set $\tilde{\mathfrak{S}}_{\mathrm{inv}}(G) = \tilde{N}(M^{\mathrm{cb}}A(G))$. Since $\tilde{N}$ is the composition of two isometries, we have that it is itself an isometry from $M^{\mathrm{cb}}A(G)$ into $\mathfrak{S}(G)$.

Lemma 5.23. *We have that $\mathcal{K}^\perp = \widetilde{\mathfrak{S}}_{\rm inv}(G)$.*

Proof. Suppose that $\varphi \in \widetilde{\mathfrak{S}}_{\rm inv}(G)$ and $h = \sum_{i=1}^k f_i \otimes g_i \in \mathcal{K}_0$. Then $m(h) = 0$, that is,

$$\int_G \sum_{i=1}^k f_i(s) g_i(ts^{-1}) ds = 0, \text{ for almost all } t \in G.$$

Let $u \in M^{\rm cb}A(G)$ be such that $\varphi = \tilde{N}(u)$. Then

$$\begin{aligned} \langle \varphi, h \rangle &= \int_{G\times G} u(st) \sum_{i=1}^k f_i(s) g_i(t) ds dt \\ &= \int_{G\times G} u(r) \sum_{i=1}^k f_i(s) g_i(s^{-1}r) ds dr \\ &= \int_{G\times G} u(r) m(h)(r) dr = 0. \end{aligned}$$

This shows that $\widetilde{\mathfrak{S}}_{\rm inv}(G) \subseteq \mathcal{K}^\perp$.

To show the reverse inclusion, fix $r \in G$ and let f_0 and g_0 be given by $f_0(x) = \Delta(r^{-1}) f(xr^{-1})$, $g_0(y) = g(ry)$, $x, y \in G$, where $f, g \in L^1(G)$. Then

$$f \otimes g - f_0 \otimes g_0 \in \mathcal{K}.$$

Indeed,

$$\begin{aligned} m(f_0 \otimes g_0)(t) &= \int_G f_0(s) g_0(s^{-1}t) ds \\ &= \int_G \Delta(r)^{-1} f_0(sr^{-1}) g(rs^{-1}t) ds \\ &= \int_G f(x) g(x^{-1}) y dx = m(f \otimes g)(t). \end{aligned}$$

Suppose that $\varphi \in \mathcal{K}^\perp$. Then

$$\begin{aligned} \int_{G\times G} \varphi(s,t) f(s) g(t) ds dt &= \int_{G\times G} \varphi(s,t) f_0(s) g_0(t) ds dt \\ &= \int_{G\times G} \varphi(s,t) \Delta(r^{-1}) f(sr^{-1}) g(rt) ds dt \\ &= \int_{G\times G} \varphi(xr, r^{-1}y) f(x) g(y) dx dy. \end{aligned}$$

It follows that $\varphi(xr, r^{-1}y) = \varphi(x, y)$ for almost all x, y. Let $\psi \in \mathfrak{S}(G)$ be given by $\psi(s,t) = \varphi(t, s^{-1})$. Then

$$\psi(sr, tr) = \varphi(tr, r^{-1}s^{-1}) = \varphi(t, s^{-1}) = \psi(s,t), \quad \text{for almost all } s, t.$$

It follows that $\psi \in \mathfrak{S}_{\rm inv}(G)$ and hence $\varphi = \tilde{\psi} \in \widetilde{\mathfrak{S}}_{\rm inv}(G)$. □

Theorem 5.24. *The space $\mathcal{Q}(G)$ is isometrically isomorphic to $(L^1(G) \otimes_h L^1(G))/\mathcal{K}$.*

Proof. Note that, by Lemma 5.23, $(L^1(G) \otimes_h L^1(G)/\mathcal{K})^*$ is isometric to $\widetilde{\mathfrak{S}}_{\rm inv}(G)$. Let $q : L^1(G) \otimes_h L^1(G) \to (L^1(G) \otimes_h L^1(G))/\mathcal{K}$ be the quotient map. For $f \in L^1(G)$ write, by virtue of the Cohen Factorisation Theorem, $f = g * h$, and set $\iota(f) = q(g \otimes h)$. The map ι is well-defined. Indeed, if $g * h = g' * h'$ for some $g', h' \in L^1(G)$, then $g \otimes h - g' \otimes h' \in \mathcal{K}_0$ and hence $q(g \otimes h) = q(g' \otimes h')$. Moreover, by the Cohen Factorisation Theorem, ι is isometric. Let $\varphi \in \widetilde{\mathfrak{S}}_{\rm inv}(G)$ and $u \in M^{\rm cb}A(G)$ be such that $\tilde{N}(u) = \varphi$. We have

$$\begin{aligned} \langle \varphi, \iota(f) \rangle &= \int_{G\times G} u(st)g(s)h(t)dsdt \\ &= \int_{G\times G} u(x)g(s)h(s^{-1}x)dsdx = \int_{G\times G} u(x)f(x)dx. \end{aligned}$$

It is easy to verify that the image of ι is dense in $(L^1(G) \otimes_h L^1(G))/\mathcal{K}$. It follows by the definition of $\mathcal{Q}(G)$ that ι extends to an isometry from $\mathcal{Q}(G)$ onto $(L^1(G) \otimes_h L^1(G))/\mathcal{K}$. □

Corollary 5.25. *The map $N : M^{\rm cb}A(G) \to \mathfrak{S}_{\rm inv}(G)$ is weak* continuous.*

Proof. Let, as in the proof of Theorem 5.24, $q : L^1(G) \otimes_h L^1(G) \to (L^1(G) \otimes_h L^1(G))/\mathcal{K}$ be the quotient map. By Theorem 5.24, $((L^1(G) \otimes_h L^1(G))/\mathcal{K})^*$ is isometric to $\mathcal{Q}(G)^* = M^{\rm cb}A(G)$; if we identify the latter two spaces, it is easily seen that $q^* : M^{\rm cb}A(G) \to \mathfrak{S}(G)$ coincides with the map $\tilde{N}$. It follows that N is weak* continuous, too. □

Exercise 5.26. *Show that the identification in Theorem 5.24 is completely isometric.*

6. Classes of multipliers

6.1. **Positive multipliers.**

Definition 6.1. *(i) Let X be a locally compact Hausdorff space. A function $k : X \times X \to \mathbb{C}$ is called* positive definite *if $(k(x_i, x_j))_{i,j=1}^n$ is a positive matrix, for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$.*

(ii) Let G be a group. A function $u : G \to \mathbb{C}$ is called positive definite *if $N(u)$ is positive definite.*

Exercise 6.2. *Let $A = (a_{i,j}) \in M_n$. The following are equivalent:*

(i) for every positive matrix $B \in M_n$, the Schur product $A * B$ of A and B is a positive matrix ;

(ii) the matrix A is positive.

We recall a version of Mercer's Theorem, which will be needed in the proof of the next proposition: if X is a locally compact Hausdorff space equipped with a Radon measure of full support and if $h \in L^2(X \times X) \cap C(X \times X)$, the integral operator T_h on $L^2(X)$ with kernel h is positive if and only if h is positive definite.

Proposition 6.3. *Let X be a σ-compact metric space, equipped with a Radon measure μ with full support. Let $k : X \times X \to \mathbb{C}$ be a continuous Schur multiplier. The following are equivalent:*
(i) S_k is positive;
(ii) k is positive definite.

Proof. (i)$\Rightarrow$(ii) By the assumption and Mercer's Theorem, kh is a positive definite function whenever $h \in L^2(X \times X) \cap C(X \times X)$ is positive definite. The statement now follows from Exercise 6.2.

(ii)$\Rightarrow$(i) follows from Mercer's Theorem, Exercise 6.2 and the fact that

$$\overline{\{T_k : k \in L^2(X \times X) \cap C(X \times X), T_k \geq 0\}}^{\|\cdot\|} = \mathcal{K}(L^2(X))^+.$$

(The latter can be seen as follows: suppose that $h \in T(X,X)$ and $\langle T_k, T_h \rangle \geq 0$ for each $k \in L^2(X \times X) \cap C(X \times X)$ with $T_k \geq 0$. By taking $k = a \otimes \bar{a}$, where $a \in C_c(G)$, we see that $(T_h a, a) \geq 0$ for all such a, and this implies that $T_h \geq 0$. It follows that $\langle T, T_h \rangle \geq 0$ for all $T \in \mathcal{K}(L^2(X))^+$, and the claim now follows from the Krein-Milman Theorem.) ☐

Exercise 6.4. *Show that the unit ball of the subspace*

$$\mathcal{A} = \left\{ \sum_{i=1}^{k} A_i T_i : A_i \in \mathcal{D}_G, T_i \in \mathrm{VN}(G) \right\}$$

is strongly dense in the unit ball of $\mathcal{B}(L^2(G))$.

Hint. Use the Stone-von Neumann Theorem, according to which the representation of the crossed product $G \times_\alpha C_0(G)$ arising from the covariant pair of representations (λ, π), where λ is the left regular representation of G and $\pi : C_0(G) \to \mathcal{B}(L^2(G))$ is given by $\pi(a) = M_a$, is faithful and its image coincides with the C*-algebra $\mathcal{K}(L^2(G))$ of all compact operators on $L^2(G)$.

Theorem 6.5. *Let G be a locally compact second countable group and $u : G \to \mathbb{C}$ be a continuous function. The following are equivalent:*
(i) $u \in M^{\mathrm{cb}}A(G)$ and S_u is completely positive;
(ii) u is positive definite.
If these conditions are fulfilled then $\|u\|_{\mathrm{cbm}} = u(e)$.

Proof. (ii)$\Rightarrow$(i) Since u is positive definite and continuous, $u \in B(G)$; by Corollary 3.9, $u \in M^{\mathrm{cb}}A(G)$. By Proposition 6.3, $S_{N(u)}$ is positive and by Exercise 4.15, $S_{N(u)}$ is completely positive. Thus, its restriction S_u to $\mathrm{VN}(G)$ is completely positive.

(i)$\Rightarrow$(ii) Since S_u is completely positive, it is also completely bounded (with $\|S_u\|_{\mathrm{cb}} = \|S_u(I)\|$) and hence $u \in M^{\mathrm{cb}}A(G)$. Let $T \in \mathcal{B}(L^2(G))$ be positive contraction and write $T = SS^*$ for some contraction $S \in \mathcal{B}(L^2(G))$.

Using Exercise 6.4, approximate S in the strong operator topology by contractions of the form $\sum_{i=1}^k A_i T_i$, where $A_i \in \mathcal{D}_G$ and $T_i \in \mathrm{VN}(G)$, $i = 1, \dots, k$. It follows that T can be approximated in the weak* topology by the operators $\sum_{i,j=1}^k A_i T_i T_j^* A_j^*$. Since S_u is completely positive and the matrix $(T_i T_j^*)_{i,j}$ is positive, $(S_u(T_i T_j^*))_{i,j}$ is positive and hence, using the fact that $S_{N(u)}$ is a $\mathcal{D}_G$-bimodule map, and letting $A = (A_1, \dots, A_k)$ be the corresonding row operator, we have

$$S_{N(u)}\left(\sum_{i,j=1}^k A_i T_i T_j^* A_j^*\right) = A(S_u(T_i T_j^*))_{i,j=1}^k A^* \geq 0.$$

By the weak* continuity of $S_{N(u)}$, we have that $S_{N(u)}$ is positive; by Proposition 6.3, u is positive definite.

The last statement follows from the fact that $S_u(I) = S_u(\lambda_e) = u(e)I$. □

The following fact, which we leave as an exercise, was established in [5].

Exercise 6.6. *The following are equivalent, for a continuous function $u : G \to \mathbb{C}$ and a natural number n:*

(i) S_u is n-positive;

(ii) for all $f_i, g_i \in C_c(G)$, $i = 1, \dots, n$, we have

$$\int_G u(s) \sum_{i=1}^n (f_i^* * f_i)(s)(g_i * \tilde{g}_i)(s) ds \geq 0.$$

In this case, $\|u\|_{\mathrm{m}} = u(e)$.

6.2. **Idempotent multipliers.** A natural class of multipliers, with importance for applications, is formed by the idempotent ones. Clearly, $u \in M^{\mathrm{cb}}A(G)$ is an idempotent Herz-Schur multiplier precisely when u is the characteristic function of a (closed and open) subset of G.

The *coset ring* of a locally compact group G is the ring of sets generated by the translates of open subgroups of G. We have the following result [27].

Theorem 6.7. *An element $u \in B(G)$ is idempotent precisely when $u = \chi_E$ for an element E of the coset ring of G.*

As a consequence of this result, note that if G is a connected group then there are no non-trivial (that is, different from 0 and 1) idempotents in $B(G)$.

This result answers completely the question of which are the idempotent Herz-Schur multipliers in the case of abelian, or more generally amenable, groups (note that in this case $B(G) = M^{\mathrm{cb}}A(G)$). The description of the idempotents in $M^{\mathrm{cb}}A(G)$ for a general group G seems to be an open question. In the positive direction, we have the following result from [49]:

Theorem 6.8. *Let G be a locally compact group and $E \subseteq G$. The following are equivalent:*

(i) $\chi_E \in M^{\mathrm{cb}}A(G)$ and $\|\chi_E\|_{\mathrm{cbm}} = 1$;

(ii) E belongs to the coset ring of G.

6.3. **Radial multipliers.** Let $r > 1$ and recall that $\mathbb{F}_r$ denotes the free group on r generators, say $a_1, \dots, a_r$. Thus, the elements of $\mathbb{F}_r$ are of the form $t = t_1 \dots t_k$, where $t_i \in \{a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1}\}$ and $t_i^{-1} \neq t_{i+1}$ for all $i = 1, \dots, k-1$ (such an expression of t is called a *reduced word*). We set $|t| = k$, and $|e| = 0$; note that $|st| \leq |s| + |t|$ and $|t^{-1}| = |t|$ We often call $|t|$ the *length* of t.

A function $\varphi : \mathbb{F}_r \to \mathbb{C}$ is called *radial* if it only depends on $|t|$; that is, if there exists a function $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) such that $\varphi(s) = \varphi(|s|)$, $s \in \mathbb{F}_r$. In the sequel, we will use the symbol $\dot{\varphi}$ to denote the function linked to a radial function φ in the above way.

Radial multipliers of $A(\mathbb{F}_r)$, that is, multipliers of $A(\mathbb{F}_r)$ which are radial functions, have been studied in great detail since U. Haagerup's paper [16], where he used them to show that the C*-algebra $C^*(\mathbb{F}_r)$ possesses the metric approximation property, although it is not nuclear. We include the following result [16], [23] which provides a source of examples of multipliers of $A(\mathbb{F}_r)$ (as we will see below, not all those are Herz-Schur multipliers, however).

Theorem 6.9. *Let $\varphi : \mathbb{F}_r \to \mathbb{C}$.*

(i) If $\sup_{s\in\mathbb{F}_r} |\varphi(s)|(1 + |s|^2) < \infty$ then $\varphi \in MA(\mathbb{F}_r)$ and

$$\|\varphi\|_{\mathrm{m}} \leq \sup_{s\in\mathbb{F}_r} |\varphi(s)|(1 + |s|^2).$$

(ii) Suppose that φ is radial. If $\sum_{n=0}^{\infty}(n+1)^2|\dot{\varphi}(n)|^2 < \infty$ then $\varphi \in MA(\mathbb{F}_r)$ and

$$\|\varphi\|_{\mathrm{m}} \leq \left(\sum_{n=0}^{\infty}(n+1)^2|\dot{\varphi}(n)|^2\right)^{\frac{1}{2}}.$$

Radial multipliers of $A(\mathbb{F}_r)$ were characterised by U. Haagerup and R. Szwarc in an unpublished manuscript [24]. Recently this characterisation was extended to the case of groups of the form

$$G = \left(*_{i=1}^{M}\mathbb{Z}_2\right) * \mathbb{F}_N, \tag{27}$$

as a consequence of much more general results on Schur multipliers on homogeneous trees.

We now explain the relation between groups of the form (27) and homogeneous trees. Let G be a discrete group, generated by a finite set, say $\mathcal{E} = \{s_1, \dots, s_n\}$, assumed to satisfy $\mathcal{E} = \mathcal{E}^{-1}$. The *Cayley graph* $\mathcal{C}_G$ of G is the graph whose vertices are the elements of G, and a (two-element) set $\{s, t\} \subseteq G$ is an edge of $\mathcal{C}_G$ if $ts^{-1} \in \mathcal{E}$. A *tree* is a connected graph without cycles. The degree of a vertex is the number of edges containing the vertex, and a graph is called *locally finite* if the degrees of all vertices are finite. It is called *homogeneous* if all vertices have the same degree (called in this case the degree of the graph).

We have the following fact [12] (see Theorem 6.3 and p. 16-18).

Theorem 6.10. *Let G be a discrete finitely generated group. The Cayley graph $\mathcal{C}_G$ of G is a locally finite homogeneous tree if and only if G is of the form (27); in this case, the degree q of $\mathcal{C}_G$ is equal to $2M + N - 1$.*

If $\mathcal{C}$ is a homogeneous tree with vertex set X, let $d(x, y)$ be the distance between two vertices x, y; that is, the length of the (unique) path connecting x and y. We set $d(x, x) = 0$. (Note that $d(x, y) = 1$ precisely when $\{x, y\}$ is an edge of $\mathcal{C}$.) Fix a vertex o of $\mathcal{C}$. A function $\varphi : X \to \mathbb{C}$ is called radial if there exists a function $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ such that $\varphi(x) = \dot{\varphi}(d(x, o))$, $x \in X$. In case $\mathcal{C} = \mathcal{C}_G$ is the Cayley graph of a group of the form (27), we choose $o = e$, the neutral element of G.

The relation between radial Herz-Schur multipliers on a group G of the form (27) and radial functions on homogeneous trees becomes clear through the following proposition.

Proposition 6.11. *Let $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ be a function, $\varphi : G \to \mathbb{C}$ be the radial function corresponding to $\dot{\varphi}$, and let $\tilde{\varphi} : G \times G \to \mathbb{C}$ be given by $\tilde{\varphi}(s, t) = \dot{\varphi}(d(s, t))$, $s, t \in G$.*

Then $\varphi \in M^{\mathrm{cb}}A(G)$ if and only if $\tilde{\varphi}$ is a Schur multiplier; in this case, $\|\varphi\|_{\mathrm{cbm}} = \|\tilde{\varphi}\|_{\mathfrak{S}}$.

Proof. Since the distance d is left invariant, we have

$$\tilde{\varphi}(s, t) = \dot{\varphi}(d(s, t)) = \dot{\varphi}(d(t^{-1}s, e)) = \varphi(t^{-1}s).$$

The claim now follows from Theorem 5.6. □

The following characterisation of radial multipliers on homogeneous trees was obtained in [23].

Theorem 6.12. *Let X be a homogeneous tree of degree $q+1$ $(2 \leq q \leq \infty)$ with distinguished vertex o. Let $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ be a function, $\varphi : X \to \mathbb{C}$ be the corresponding radial function and $\tilde{\varphi}(x, y) = \dot{\varphi}(d(x, y))$, $x, y \in X$. Set $H = (h_{i,j})_{i,j \in \mathbb{N}_0}$, where*

$$h_{i,j} = \dot{\varphi}(i + j) - \dot{\varphi}(i + j + 2), \quad i, j \in \mathbb{N}_0.$$

(i) The function $\tilde{\varphi}$ is a Schur multiplier if and only if H is the matrix of a trace class operator.

(ii) (For simplicity we assume that $q = \infty$). If the statements in (i) hold, then the limits

$$\lim_{n\to\infty} \dot{\varphi}(2n) \text{ and } \lim_{n\to\infty} \dot{\varphi}(2n + 1)$$

exist. Setting

$$c_{\pm} = \frac{1}{2}(\lim_{n\to\infty} \dot{\varphi}(2n) \pm \lim_{n\to\infty} \dot{\varphi}(2n + 1)),$$

we have that

$$\|\tilde{\varphi}\|_{\mathfrak{S}} = |c_+| + |c_-| + \|H\|_1.$$

Theorem 6.12 can be used [23] to establish the following result. We note that, in case G is a non-commutative free group, the result was obtained by M. Bożejko in [2].

Theorem 6.13. *Let G be a group of the form (27). There exists a radial function φ which lies in $MA(G)$ but not in $M^{\rm cb}A(G)$.*

Idea of proof. Let φ be the radial function associated with the function $\dot\varphi : \mathbb{N}_0 \to \mathbb{C}$ given by $\dot\varphi(n) = 0$ if $n \neq 2^k$, $k \in \mathbb{N}$, and $\dot\varphi(2^k) = \frac{1}{k2^k}$, $k \in \mathbb{N}$. Then

$$\sum_{n=0}^{\infty}(n+1)^2|\dot\varphi(n)|^2 < \infty,$$

which by Theorem 6.9 implies that $\varphi \in MA(G)$. One can now show directly that the corresponding matrix H does not belong to the trace class.

There is a natural version of radiality that involves the free product of arbitrary groups. Let G_i, $i = 1, \dots, n$, be discrete groups of the same cardinality (either finite or countably infinite), and set $G = *_{i=1}^n G_i$. Every element g of G has a unique representation $g = g_{i_1}g_{i_2}\cdots g_{i_k}$, where $g_{i_m} \in G_{i_m}$ are distinct from the corresponding neutral elements and $i_1 \neq i_2 \neq \cdots \neq i_m$. The number m is called the *block length* of g and denoted $\|g\|$. Call a function $\varphi : G \to \mathbb{C}$ in this setting radial if it depends only on $\|g\|$. The following result of J. Wysoczański [51] should be compared to Haagerup's characterisation of radial multipliers on $\mathbb{F}_n$. We note that explicit formulas for the corresponding multiplier norm are given in [51].

Theorem 6.14. *Let $\dot\varphi : \mathbb{N}_0 \to \mathbb{C}$ and $\varphi : G \to \mathbb{C}$ be the corresponding radial function with respect to the block length. The following are equivalent:*

(i) $\varphi \in M^{\rm cb}A(G)$;

(ii) the matrix $(\dot\varphi(i+j) - \dot\varphi(i+j+1))_{i,j}$ defines a trace class operator on $\ell^2(\mathbb{N}_0)$.

We now turn our attention to the completely positive radial multipliers on $\mathbb{F}_n$. The following result is taken from [16].

Theorem 6.15. *Let $0 < \theta < 1$. Then the function $t \to \theta^{|t|}$ on $\mathbb{F}_n$ or on $\mathbb{F}_\infty$ is positive definite.*

Proof. We give only a sketch of the proof. It uses Shoenberg's Theorem, according to which, if $k(x,y)$ is a conditionally negative definite kernel, then $e^{-k(x,y)}$ is a positive definite function. The kernel $k : X \times X \to \mathbb{R}$ being conditionally negative definite means the following: $k(x,x) = 0$ for all x, $k(x,y) = k(y,x)$ for all x, y, and

$$\sum_{i,j=1}^{m} k(x_i,x_j)\alpha_i\alpha_j \leq 0,$$

for all $x_1, \dots, x_m \in X$ and all $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 0$. It is known that k is conditionally negative definite if and only if there exists a function $b : X \to H$, where H is a Hilbert space, such that $k(x,y) = \|b(x) - b(y)\|$, $x, y \in X$.

Hence, in order to establish the theorem, it suffices to show that the kernel k, given by $k(s,t) = |s^{-1}t|$, is conditionally negative definite. This is done by exhibiting a Hilbert space H and a function $b : \mathbb{F}_n \to H$ such that

$$|s^{-1}t| = \|b(s) - b(t)\|. \tag{28}$$

Fix generators $a_1, \dots, a_n$ of $\mathbb{F}_n$. The Hilbert space H can be taken to be $\ell^2(\Lambda)$, where $\Lambda = \{(s,t) \in \mathbb{F}_n \times \mathbb{F}_n : s^{-1}t = a_i, \text{ for some } i\}$. Let $\{e_{(s,t)} : (s,t) \in \Lambda\}$ be the corresponding orthonormal basis of H. If $s^{-1}t = a_i^{-1}$ for some i, then set $e_{(s,t)} = -e_{(t,s)}$. For an element $s = s_1 s_2 \dots s_k$, where s_i is either a generator or its inverse, let

$$b(s) = e_{(e,s_1)} + e_{(s_1,s_1s_2)} + \dots + e_{(s_1s_2\dots s_{k-1},s)}.$$

We leave it as an exercise to show that identity (28) holds with this choice of H and b. □

Equipped with Theorem 6.15, it is now easy to establish the following characterisation.

Exercise 6.16. *Let $\theta \in \mathbb{R}$. Then the function $\varphi_\theta : t \to \theta^{|t|}$ on $\mathbb{F}_n$ is positive definite if and only if $-1 \le \theta \le 1$.*

It turns out that the functions φ_θ can be used to synthesise all positive definite radial functions on $\mathbb{F}_\infty$: the following characterisation was obtained by U. Haagerup and S. Knudby in [21]:

Theorem 6.17. *Let $\varphi : \mathbb{F}_\infty \to \mathbb{C}$ be a radial function with $\varphi(e) = 1$. The following are equivalent:*
(i) The function φ is positive definite;
(ii) There exists a probability measure μ on $[-1,1]$ such that

$$\varphi(x) = \int_{-1}^{1} \theta^{|x|} d\mu(\theta), \quad x \in \mathbb{F}_n.$$

Moreover, if (ii) holds, then μ is uniquely determined by φ.

In the case of $\mathbb{F}_r$, $1 < r < \infty$, the general form of positive definite radial multipliers is different. The rest of the section is dedicated to the treatment of that case; the material is taken from U. Haagerup and S. Knudby's paper [21] and the monograph [13]. For the purpose of motivation, we start with suggesting the following exercise.

Exercise 6.18. *Let $u : \mathbb{Z} \to \mathbb{R}$ be a symmetric function, that is, $u(n) = u(-n)$ for each $n \in \mathbb{Z}$. Show that there exists a finite positive Borel measure μ on $[0, \pi]$ such that*

$$u(n) = \int_0^\pi \cos(n\theta) d\mu(\theta), \quad n \in \mathbb{N}.$$

Let $E_n = \{x \in \mathbb{F}_r : |x| = n\}$. For $n > 0$, let μ_n be the function taking the same constant value on the elements of E_n and zero on $\mathbb{F}_r \setminus E_n$, such that $\sum_x \mu_n(x) = 1$ (note that the constant value equals $\frac{1}{2r(2r-1)^{n-1}}$). Let also μ_0 be the characteristic function of the singleton $\{e\}$. Denote by $\mathcal{A}$ the subalgebra of the group algebra $\mathbb{C}[\mathbb{F}_r]$ generated by μ_n, $n \geq 0$ – this is the algebra of all radial functions on $\mathbb{F}_r$, equipped with the operation of convolution. Clearly, $\mathcal{A}$ is the linear span of $\{\mu_n : n \geq 0\}$.

Lemma 6.19. *Let $q = 2r - 1$. Then*

$$\mu_1 * \mu_n = \frac{1}{q+1}\mu_{n-1} + \frac{q}{q+1}\mu_{n+1}, \quad n \geq 1.$$

Proof. We have

$$\mu_1 * \mu_n(x) = \sum_{y \in \mathbb{F}_r} \mu_1(y)\mu_n(y^{-1}x) = \frac{1}{q+1}\sum_{|y|=1} \mu_n(y^{-1}x). \tag{29}$$

Let $\{a_1, \dots, a_{q+1}\}$ be the set of words of length one. If $|x| = n+1$, then among the words $a_j x$, $j = 1, \dots, q$, there is only one of length n, namely, the word $a_j x$ for which $x = a_j^{-1}x'$ (for some $x' \in \mathbb{F}_r$). Thus, in this case

$$\mu_1 * \mu_n(x) = \frac{1}{q+1}\frac{1}{(q+1)q^{n-1}} = \frac{q}{q+1}\mu_{n+1}(x).$$

If $|x| = n-1$, then among the words $a_j x$, $j = 1, \dots, r$, there are q of length n, and thus

$$\mu_1 * \mu_n(x) = \frac{1}{q+1}\frac{q}{(q+1)q^{n-1}} = \frac{1}{q+1}\mu_{n-1}(x).$$

Finally, if x has length different from $n+1$ or $n-1$ then all words $a_j x$, $j = 1, \dots, q$ have length different from n and hence the right hand side of (29) is zero. The claim follows. □

Define a sequence (P_n) of polynomials by setting $P_0(x) = 1$, $P_1(x) = x$ and

$$P_{n+1}(x) = \frac{q+1}{q}xP_n(x) - \frac{1}{q}P_{n-1}(x), \quad n \geq 1. \tag{30}$$

By the definition of this sequence, we have that

$$\mu_n = P_n(\mu_1), \quad n \geq 0. \tag{31}$$

(Here, the product is taken with respect to the convolution.)

The *Laplace operator* is the (linear) map L acting on $\mathbb{C}[\mathbb{F}_r]$ and given by

$$L\varphi = \mu_1 * \varphi, \quad \varphi \in \mathbb{C}[\mathbb{F}_r]\,. \tag{32}$$

It is clear that, if $\varphi \in \mathbb{C}[\mathbb{F}_r]$ and $x \in \mathbb{F}_r$, then

$$L\varphi(x) = \frac{1}{q+1}\sum_y \varphi(y), \tag{33}$$

where the sum is taken over all neighbours y of x in the the Cayley tree of $\mathbb{F}_r$.

Spherical functions, defined now, were first introduced in [13].

Definition 6.20. *Call a function $\varphi \in \mathbb{C}[\mathbb{F}_r]$ spherical if φ is radial, $\varphi(e) = 1$ and $L\varphi = s\varphi$ for some $s \in \mathbb{C}$.*

Thus, spherical functions are the normalised radial eigenvalues of the Laplace operator. Suppose that $\varphi \in \mathbb{C}[\mathbb{F}_r]$ is spherical and let $\dot{\varphi}$ be as usual the underlying function defined on $\mathbb{N}_0$. Identity (33) implies

$$\dot{\varphi}(0) = 1,\ \dot{\varphi}(1) = s,\ \dot{\varphi}(n+1) = \frac{q+1}{q} s\dot{\varphi}(n) - \frac{1}{q}\dot{\varphi}(n-1). \tag{34}$$

Using (30), we now see that

$$\dot{\varphi}(n) = P_n(s), \quad n \geq 0. \tag{35}$$

It also follows that for each $s \in \mathbb{C}$ there exists a unique spherical function corresponding to the eigenvalue s; we denote this function by φ_s.

On the group algebra $\mathbb{C}[\mathbb{F}_r]$, consider the bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \sum_{x \in \mathbb{F}_r} f(x)g(x), \quad f, g \in \mathbb{C}[\mathbb{F}_r]\,.$$

If $\varphi \in \mathcal{A}$ then

$$\langle \mu_n, \varphi \rangle = \dot{\varphi}(n), \quad n \geq 0.$$

Thus, by (35),

$$\langle \mu_n, \varphi_s \rangle = P_n(s), \quad n \geq 0. \tag{36}$$

We have the following characterisation of spherical functions.

Lemma 6.21. *Let $\varphi : \mathbb{F}_r \to \mathbb{C}$ be a non-zero radial function. The following are equivalent:*

(i) φ is spherical;

(ii) the functional $f \to \langle f, \varphi \rangle$ on $\mathcal{A}$ is multiplicative.

Proof. (i)⇒(ii) Let $s \in \mathbb{C}$. By (31) and (36),

$$\langle P_n(\mu_1), \varphi_s \rangle = P_n(s), \quad n \geq 0.$$

The set $\{P_n : n \geq 0\}$ spans the set of all polynomials, and hence by linearlity

$$\langle P(\mu_1), \varphi_s \rangle = P(s), \qquad P \text{ a polynomial.}$$

On the other hand, the map $P \to P(\mu_1)$, is a bijective algebra homomorphism from the algebra of all polynomials onto $\mathcal{A}$. Statement (ii) now follows.

(ii)⇒(i) For $n > 0$ we have

$$\langle \mu_n, \varphi \rangle = \langle \mu_0 * \mu_n, \varphi \rangle = \langle \mu_0, \varphi \rangle \langle \mu_n, \varphi \rangle$$

and hence $\dot{\varphi}(0) = \langle \mu_0, \varphi \rangle = 1$.

Let $s = \dot{\varphi}(1) = \langle \mu_1, \varphi \rangle$. Then

$$\langle \mu_1 * \mu_n, \varphi \rangle = \langle \mu_1, \varphi \rangle \langle \mu_n, \varphi \rangle = s\dot{\varphi}(n).$$

On the other hand, by Lemma 6.19,

$$\begin{aligned}\langle \mu_1 * \mu_n, \varphi\rangle &= \left\langle \frac{1}{q+1}\mu_{n-1}, \varphi\right\rangle + \left\langle \frac{q}{q+1}\mu_{n+1}, \varphi\right\rangle \\ &= \frac{1}{q+1}\dot{\varphi}(n-1) + \frac{q}{q+1}\dot{\varphi}(n+1).\end{aligned}$$

Thus, (34) holds for φ and hence $\varphi = \varphi_s$. $\square$

Let $\mathcal{E} : \mathbb{C}[\mathbb{F}_r] \to \mathcal{A}$ be the map given by

$$\mathcal{E}(f)(x) = \frac{1}{(q+1)q^{n-1}} \sum_{|y|=n} f(y), \quad |x| = n;$$

thus,

$$\mathcal{E}(f)(x) = \langle f, \mu_n\rangle, \quad |x| = n.$$

Lemma 6.22. *(i) The following two properties hold:*
(a) $\mathcal{E}(f) = f$ if $f \in \mathcal{A}$;
(b) $\langle f, \mathcal{E}(g)\rangle = \langle f, g\rangle$ if f is radial.
Moreover, if $\mathcal{E}' : \mathbb{C}[\mathbb{F}_r] \to \mathcal{A}$ satisfies (a) and (b) then $\mathcal{E}' = \mathcal{E}$.
(ii) Let $\mathcal{R}$ be the von Neumann subalgebra of $\mathrm{VN}(\mathbb{F}_r)$ generated by $\mathcal{A}$. Then the map $\mathcal{E}$ extends to a normal conditional expectation from $\mathrm{VN}(\mathbb{F}_r)$ onto $\mathcal{R}$.

Proof. (i) Properties (a) and (b) are straightforward. Suppose $\mathcal{E}' : \mathbb{C}[\mathbb{F}_r] \to \mathcal{A}$ satisfies (a) and (b). If $f \in \mathbb{C}[\mathbb{F}_r]$ and $x \in \mathbb{F}_r$ then

$$\mathcal{E}'(f)(x) = \langle \mathcal{E}'(f), \delta_x\rangle = \langle \mathcal{E}'(f), \mathcal{E}(\delta_x)\rangle = \langle f, \mathcal{E}(\delta_x)\rangle = \mathcal{E}(f)(x).$$

(ii) By general von Neumann algebra theory, there exists a normal conditional expectation from $\mathrm{VN}(\mathbb{F}_r)$ onto $\mathcal{R}$. Its restriction on $\mathbb{C}[\mathbb{F}_r]$ must satisfy (a) and (b), and by (i) it must coincide with $\mathcal{E}$. $\square$

Proposition 6.23. *The function φ_s is positive definite if and only if $-1 \leq s \leq 1$.*

Proof. Suppose that $-1 \leq s \leq 1$. By (34), φ_s is real-valued. It was shown in [13] that in this case φ_s is also bounded. Let $\overline{\mathcal{A}}$ be the closure of $\mathcal{A}$ in $\ell^1(\mathbb{F}_r)$. Since φ_s is radial, we have that $\varphi_s(x) = \varphi_s(x^{-1})$ for all $x \in \mathbb{F}_r$. We claim that the functional $f \to \langle f, \varphi_s\rangle$ on $\overline{\mathcal{A}}$ is positive. Indeed, if $f \in \overline{\mathcal{A}}$ then, using Lemma 6.21, we have

$$\begin{aligned}\langle f * f^*, \varphi_s\rangle &= \langle f, \varphi_s\rangle\langle f^*, \varphi_s\rangle = \left(\sum_{x\in\mathbb{F}_r} f(x)\varphi_s(x)\right)\left(\sum_{x\in\mathbb{F}_r} \overline{f(x^{-1})}\varphi_s(x)\right) \\ &= \left(\sum_{x\in\mathbb{F}_r} f(x)\varphi_s(x)\right)\left(\sum_{x\in\mathbb{F}_r} \overline{f(x^{-1})\varphi_s(x^{-1})}\right) \\ &= \langle f, \varphi_s\rangle\overline{\langle f, \varphi_s\rangle} \geq 0.\end{aligned}$$

Now let $f \in \ell^1(\mathbb{F}_r)$ be positive. Then $\mathcal{E}(f)$ is positive and by Lemma 6.22 and the previous paragraph,

$$\langle f, \varphi_s \rangle = \langle \mathcal{E}(f), \varphi_s \rangle \geq 0.$$

It follows that the functional on $\ell^1(\mathbb{F}_r)$, $f \to \langle f, \varphi_s \rangle$, is positive, and hence φ_s is positive definite.

Conversely, suppose that φ_s is positive definite. Then $\overline{\varphi_s(x)} = \varphi_s(x^{-1})$ and since $|x| = |x^{-1}|$, the function φ_s is real-valued. Since φ_s is also bounded, we have, by (35), that $-1 \leq s \leq 1$. □

Theorem 6.24. *Let $\varphi : \mathbb{F}_r \to \mathbb{C}$ be a radial function with $\varphi(e) = 1$. The following are equivalent:*

(i) φ is positive definite;

(ii) there exists a probability measure μ on $[-1, 1]$ such that

$$\varphi(x) = \int_{-1}^{1} \varphi_s(x) d\mu(s), \qquad x \in \mathbb{F}_r.$$

If (ii) holds true then the measure μ is uniquely determined by φ.

Proof. (ii)⇒(i) follows from Proposition 6.23.

(i)⇒(ii) Let Φ (resp. Φ_s, $-1 \leq s \leq 1$) be the state on $C^*(\mathbb{F}_r)$ which corresponds to φ (resp. φ_s, $-1 \leq s \leq 1$) on $\mathbb{C}[\mathbb{F}_r]$. Let $C^*(\mu_1)$ be the C*-subalgebra of $C^*(\mathbb{F}_r)$ generated by μ_1; since $\mathcal{A}$ is generated by μ_1 as an algebra, $C^*(\mu_1)$ coincides with the closure of $\mathcal{A}$ in $C^*(\mathbb{F}_r)$. We have that $\mu_1 = \mu_1^*$ and $\|\mu_1\| \leq 1$ (indeed, in every representation of $\mathbb{F}_r$, the image of μ_1 is the average of r unitary operators and hence has norm at most 1), we have that the spectrum of μ_1 is contained in $[-1, 1]$. Conversely, since $\Phi_s(\mu_1) = \langle \mu_1, \varphi_s \rangle = s$, we have that the spectrum of μ_1 coincides with $[-1, 1]$.

It follows that $C^*(\mu_1)$ is *-isomorphic to $C([-1, 1])$. The restriction of Φ to $C^*(\mu_1)$ hence yields a state on $C([-1, 1])$; by the Riesz Representation Theorem, there exists a probability measure μ on $[-1, 1]$ such that

$$\Phi(f(\mu_1)) = \int_{-1}^{1} f(s) d\mu(s), \quad f \in C([-1, 1]).$$

Now taking $f = P_n$, we obtain

$$\dot{\varphi}(n) = \Phi(\mu_n) = \Phi(P_n(\mu_1)) = \int_{-1}^{1} P_n(s) d\mu(s) = \int_{-1}^{1} \dot{\varphi}_s(n) d\mu(s).$$

The uniqueness follow from the fact that the polynomials P_n span all polynomials; the details are left as an exercise. □

Radial multipliers, considered in this section, have had a number of applications; among them is the construction of various approximate identities in relation with approximation properties for locally compact groups. We consider some of those in the next section.

7. Approximation properties for groups

In this brief section, we introduce two more approximation properties for groups. They are defined in terms of Herz-Schur multipliers, and have important counterparts as approximation properties of operator algebras. The link between the two is given by passing from a group G to the C*-algebra $C_r^*(G)$ or the von Neumann algebra $\mathrm{VN}(G)$. However, we will not discuss this relation.

We first recall that a locally compact group G is *amenable* if $A(G)$ possesses a bounded approximate identity. It is known that G is amenable if and only if there exists a net (u_i) of continuous compactly supported positive definite functions such that $u_i \to 1$ uniformly on compact sets.

Amenability is a fairly restrictive property and in some cases weaker approximation properties prove to be more instrumental. Such is the property of weak amenability, first defined by M. Cowling and U. Haagerup in [7].

Definition 7.1. [7] *A locally compact group G is called* weakly amenable *if there exists a net $(u_i) \subseteq A(G)$ and a constant $C > 0$ such that $\|u_i\|_{\mathrm{cbm}} \leq C$ and $u_i \to 1$ uniformly on compact sets.*

If G is weakly amenable, the infimum of all constants C appearing in Definition 7.1 is denoted by Λ_G. It was shown in [7, Proposition 1.1] that if G is a weakly amenable group then the net (u_i) from Definition 7.1 can moreover be chosen so that the following conditions are satisfied:

- u_i is compactly supported for each i;
- $u_i u \to u$ in the norm of $A(G)$, for every $u \in A(G)$.

It is easy to see that every amenable group is weakly amenable.

The notion of weak amenability has been studied extensively; one of the first results in this direction was the fact that $\mathbb{F}_n$ is weakly amenable [16]. The multipliers that were utilised in this setting were radial. Since non-commutative free groups are not amenable, we have that the class of weakly amenable groups is strictly larger than that of amenable ones.

The weak amenability of $\mathbb{F}_n$ was generalised in [4] by showing the following:

Theorem 7.2. *Let G_i, $i \in I$, be amenable locally compact groups, each of which contains a given open compact group H. Then the free product G of the family $(G_i)_{i\in I}$ over H is weakly amenable and $\Lambda_G = 1$.*

The multipliers that are utilised in establishing the latter result were also radial.

We point out a functoriality property of weak amenability: if G_1 and G_2 are discrete groups then $\Lambda_{G_1\times G_2} = \Lambda_{G_1}\Lambda_{G_2}$.

An even weaker approximation property for groups was introduced by U. Haagerup and J. Kraus in [22]. We refer to Section 5.4 for the weak* topology used in the definition below.

Definition 7.3. *A locally compact group G is said to have the* approximation property (AP) *if there exists a net $(u_i) \subseteq A(G)$ such that $u_i \to 1$ in the weak* topology of $M^{\rm cb}A(G)$.*

Exercise 7.4. *(i) Show that the functions u_i from Definition 7.3 can be chosen of compact support.*

(ii) Show that every weakly amenable locally compact group has the approximation property.

The following result was established in [22].

Theorem 7.5. *The following are equivalent, for a locally compact group G:*

(i) G has (AP);

(ii) for every locally compact group H, there exists a net $(u_i) \subseteq A(G)$ of functions with compact support such that $(u_i \otimes 1)$ is an approximate identity for $A(G \times H)$;

(iii) there exists a net $(u_i) \subseteq A(G)$ of functions with compact support such that $(u_i \otimes 1)$ is an approximate identity for $A(G \times SU(2))$.

The difference between amenability, weak amenability and (AP) are clearly highlighted in the following result.

Theorem 7.6. *Let G be a locally compact group.*

(i) the group G is weakly amenable with $\Lambda_G \leq L$ if and only if the constant function 1 can be approximated in the weak topology of $M^{\rm cb}A(G)$ by elements of the set $\{u \in A(G) : \|u\|_{\rm cbm} \leq L\}$.*

(ii) the group G is amenable if and only if the constant function 1 can be approximated in the weak topology of $M^{\rm cb}A(G)$ by elements of the set $\{u \in A(G) : u \text{ positive definite}, u(e) = 1\}$.*

The approximation property has the following nice functoriality property [21]:

Theorem 7.7. *Let G be a locally compact group and H be a closed normal subgroup of G. If H and G/H have (AP) then so does G.*

We include an (incomplete) list of examples of groups in relation with weak amenability and (AP).

- [5] $SO_o(n,1)$: the connected component of the identity of the group $SO(n,1)$ of all real $(n+1)\times(n+1)$ matrices with determinant 1, leaving the quadratic form $-t_0^1 + t_1^2 + \cdots + t_n^2$ invariant. Here $\Lambda_{SO_o(1,n)} = 1$.
- [7] More generally, connected real Lie groups with finite centre that are locally isomorphic to $SO(1,n)$ or $SU(1,n)$. Here $\Lambda_G = 1$. (finiteness of centre removed in [25]).
- More generally, real simple Lie groups of real rank one are weakly amenable ([5], [7], [25]).
- [19] Real simple Lie groups of real rank at least two are not weakly amenable.

- [38] Hyperbolic groups are weakly amenable.
- [39] Wreath products of arbitrary groups by non-amenable groups are not weakly amenable.
- [20] Connected simple Lie groups with finite centre and real rank at least two do not have the (AP).
- $SL(2, Z) \rtimes Z_2$ has (AP) but is not weakly amenable [21], [19].
- [32] $SL(3, Z)$ does not have (AP).

A characterisation of weak amenability for connected Lie groups was given in [6].

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Pure Mathematics Research Centre, Queen's University Belfast, Belfast BT7 1NN, United Kingdom

E-mail address: i.todorov@qub.ac.uk