An uncertainty principle for unimodular quantum groups

Jason Crann
with Mehrdad Kalantar

Carleton University and Université Lille 1

Workshop on Operator Spaces, Locally Compact Quantum Groups and Amenability
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Outline

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Background

Main Result

Non-unimodular Setting

Position and momentum operators on $L_2(\mathbb{R})$:

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Uncertainty relation:

$$\sigma(Q, f)\sigma(P, f) \geq \frac{\hbar}{2},$$

where $\sigma(Q, f) = \sqrt{\langle (Q - \langle Qf, f\rangle)^2 f, f\rangle}$, and $\|f\|_2 = 1$.

Q and P are unitarily equivalent via **Fourier transform** ($\hbar \equiv 1$):

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx.$$

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Stronger relation using **entropy**:

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Theorem (Hirschman '57; Beckner '75)

For $f \in L_2(\mathbb{R})$ such that $\|f\|_2 = 1$, we have

$$H(|f|^2) + H(|\mathcal{F}(f)|^2) \geq \log(\pi e)$$

whenever the LHS is defined.

Fact: $H(|f|^2) + H(|\mathcal{F}(f)|^2) \geq \log(\pi e) \Rightarrow \sigma(Q, f)\sigma(P, f) \geq \frac{\hbar}{2}$

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Theorem (Hirschman '57)

Let G be a locally compact Abelian group. Then for $f \in L_2(G)$ such that $\|f\|_2 = 1$, we have

$$H(|f|^2) + H(|\mathcal{F}(f)|^2) \geq 0$$

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Remark: For compact Abelian groups, $\exists$ f such that $H(|f|^2) + H(|\mathcal{F}(f)|^2) = 0$.

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Question: Manifestation in non-Abelian group duality?

Outline

Definition (Kustermans–Vaes '00)

A ***LCQG*** $\mathbb{G} = (M, \Gamma, \varphi, \psi)$

- *M is a* ***von Neumann algebra****;*
- $\Gamma : M \to M \bar{\otimes} M$ *is a* ***co-multiplication****.*
- *φ is a* ***left Haar weight*** *on M:*

$$\varphi((\omega \otimes \iota)\Gamma(x)) = \omega(1)\varphi(x), \quad x \in \mathcal{M}_\varphi, \ \omega \in M_*;$$

- *ψ is a* ***right Haar weight*** *on M:*

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Notation: $L_\infty(\mathbb{G}) := M, \ L_1(\mathbb{G}) := M_*, \ L_2(\mathbb{G}) := L_2(M, \varphi)$.

$\mathbb{G}$ is **unimodular** if $\varphi = \psi$ is tracial (i.e., $\mathbb{G}$ is a unimodular Kac algebra).

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In this case, $\forall$ state $f \in L_1(\mathbb{G})$ there exists a **density** $D \; \eta \; L_\infty(\mathbb{G})$ such that

$$\langle f, x \rangle = \varphi(Dx), \quad x \in L_\infty(\mathbb{G}).$$

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If $D = \int_0^\infty \lambda de_\lambda$, then we define the **entropy** of f by

$$H(f) := H(D) = -\varphi(D \log D) = -\int_0^\infty \lambda \log \lambda d\varphi(e_\lambda).$$

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Ex: If $\mathbb{G}_a = (L_\infty(G), \Gamma_a, \varphi_a = \psi_a)$, and $f \in L_1(G)$ is a state, then $D = M_f$ and

$$H(f) = -\int_G f(s) \log f(s) ds.$$

Non-commutative Fourier transform $\mathcal{F} : L_2(\mathbb{G}) \to L_2(\hat{\mathbb{G}})$ is the unique extension of the map

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Given a state $\rho \in \mathcal{T}(L_2(\mathbb{G}))$, let D_ρ be the **density** of $\rho|_{L_\infty(\mathbb{G})}$, and $\hat{D}_\rho$ be the **density** of $\mathcal{F}\rho\mathcal{F}^*|_{L_\infty(\hat{\mathbb{G}})}$.

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Theorem (C–Kalantar '14)

Let $\mathbb{G}$ be a ***unimodular*** *LCQG, and $\rho \in \mathcal{T}(L_2(\mathbb{G}))$ be a state. Then if $H(D_\rho)$, $\hat{H}(\hat{D}_\rho)$, $S_{vN}(\rho)$ are finite,*

$$H(D_\rho) + \hat{H}(\hat{D}_\rho) \geq S_{vN}(\rho).$$

In particular, if $\rho = \omega_\xi$, then $H(D_\xi) + \hat{H}(\hat{D}_\xi) \geq 0$.

Lemma (Gibbs Variational Principle)

Let $A \in \mathcal{L}(H)$ be self-adjoint s.t. $\mathrm{tr}(e^{-A}) < \infty$. Then $\forall$ state $\rho \in \mathcal{T}(H)$

$$\mathrm{tr}(\rho A) + \mathrm{tr}(\rho \log \rho) \geq -\log \mathrm{tr}(e^{-A})$$

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Lemma (Golden–Thompson Inequality)

Let $A, B \in \mathcal{L}(H)$ *be self-adjoint operators bounded from above, then*

$$\mathrm{tr}(e^{A+B}) \leq \mathrm{tr}(e^{A/2} e^{B} e^{A/2}).$$

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Lemma (C–Kalantar '14)

Let $\mathbb{G}$ *be a Kac algebra and let* $f \in L_1(\mathbb{G})^+$. *Then* $\exists$ *a net* $(\hat{a}_k)$ *in* $L_\infty(\hat{\mathbb{G}})$ *s.t.* $\sum_{k\in K} \hat{a}_k^* \hat{a}_k = \sum_{k\in K} \hat{a}_k \hat{a}_k^* = \langle f, 1\rangle 1$, *and*

$$\Theta(f)(T) := (f \otimes \iota)W^*(1 \otimes T)W = \sum_{k\in K} \hat{a}_k^* T \hat{a}_k, \quad T \in \mathcal{B}(L_2(\mathbb{G})).$$

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If $D_\rho = \int_0^\infty \lambda de_\lambda$ and $\hat{D}_\rho = \int_0^\infty \lambda d\hat{e}_\lambda$, their t^{th} **singular numbers**:

$\mu_t(D_\rho) = \inf\{s \geq 0 \mid \varphi(e_{(s,\infty)}) \leq t\}$ and

$\hat{\mu}_t(\hat{D}_\rho) = \inf\{s \geq 0 \mid \hat{\varphi}(\hat{e}_{(s,\infty)}) \leq t\}$, for $t > 0$.

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$\rightsquigarrow$ **probability densities** on $(0, \infty)$ satisfying

$H(D_\rho) = -\int_0^\infty \mu_t(D_\rho) \log \mu_t(D_\rho) dt$

$\hat{H}(\hat{D}_\rho) = -\int_0^\infty \hat{\mu}_t(\hat{D}_\rho) \log \hat{\mu}_t(\hat{D}_\rho) dt$

where dt is the Lebesgue measure (*Fack–Kosaki '86*).

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$$H(\mu) + \hat{H}(\hat{\mu}) \geq S_{vN}(\rho)$$

Compact Case

Theorem (Matolcsi–Szücs '73)

Let G be compact group and let $f \in L_2(G)$, $f \neq 0$. Then

$$h_G(supp(f))\Bigg(\sum_{\pi | \hat{f}(\pi) \neq 0} d\pi^2 \Bigg) \geq 1,$$

where $\hat{f}(\pi) = \int_G f(s)\pi(s)^* ds$, $\pi \in \hat{G}$.

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Theorem (Alagic–Russell '09)

Let G be compact group and let $f \in L_2(G)$, $f \neq 0$. Then

$$h_G(supp(f))\Big(\sum_{\pi \mid \hat{f}(\pi)\neq 0} d\pi \cdot rank(\hat{f}(\pi))\Big) \geq 1.$$

Compact Case

If $\mathbb{G}$ is a compact Kac algebra, $A := \mathrm{Irred}(\mathbb{G})$, then

$$\mathcal{F} : L_2(\mathbb{G}) \ni \xi \to \bigoplus_{\alpha \in A} \alpha(b(\xi)) \in \ell^2 - \bigoplus_{\alpha \in A} \mathcal{HS}(H_\alpha)$$

where $b : L_2(\mathbb{G}) \to L_1(\mathbb{G})$ and $\alpha(f) = (f \otimes \iota)(u^\alpha)$.

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Corollary (C–Kalantar '14)

Let $\mathbb{G}$ be a compact Kac algebra, and $\rho \in \mathcal{T}(L_2(\mathbb{G}))^+ - \{0\}$. Then

$$\varphi(s(D_\rho)) \left(\sum_{\alpha \in A} d_\alpha \cdot \mathrm{rank}(\hat{D}^\alpha) \right) \geq e^{S_{vN}\left(\frac{\rho}{\mathrm{tr}(\rho)}\right)}$$

where $s(D_\rho) = supp(\rho|_{L_\infty(\mathbb{G})})$. In particular, for $\xi \in L_2(\mathbb{G}) - \{0\}$,

$$\varphi(s(\omega_\xi)) \left(\sum_{\alpha \in A} d_\alpha \cdot \mathrm{rank}(\alpha(b(\xi))) \right) \geq 1$$

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Observation: If $\mathbb{G}$ is unimodular, $f \in L_1(\mathbb{G})_1^+$, then

$$H(f) = -S(f, \varphi), \text{ the } \textbf{relative entropy}.$$

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Let $M \subseteq \mathcal{B}(H)$, and let ψ be a n.s.f. weight on M'. Then

$$\mathcal{D}(H, \psi) := \{\xi \in H \mid \exists R_\xi \in \mathcal{B}(H_\psi, H) \text{ s.t. } R_\xi \Lambda_\psi(x') = x'\xi\}$$

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It follows that $R_\xi R_\xi^* \in M$ for all $\xi \in \mathcal{D}(H, \psi)$.

For any normal semi-finite weight φ on M, the **spatial derivative** $d\varphi/d\psi$ is the largest positive self-adjoint operator $T \in \mathcal{L}(H)$ s.t.

$$\varphi(R_\xi R_\xi^*) = \begin{cases} \left\| T^{1/2}\xi \right\|^2 & \text{if } \xi \in \mathcal{D}(H, \psi) \cap \mathcal{D}(T^{1/2}), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ be an arbitrary LCQG. Then any state $f \in L_1(\mathbb{G})$ satisfies $f = \omega_\xi|_{L_\infty(\mathbb{G})}$. We define the **entropy of f** by

$$H(f) := -S(f, \varphi) = -\left\langle \log\left(\frac{d\omega'_\xi}{d\varphi}\right)\xi, \xi \right\rangle,$$

where $\omega'_\xi = \omega_\xi|_{L_\infty(\mathbb{G})'}$. Definition is independent of the representing vector ξ.

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$$H(\omega_\xi) = H'(\omega'_{J\xi}) = -\Big\langle \log\Big(\frac{d\omega_{J\xi}}{d\varphi'}\Big)J\xi, J\xi\Big\rangle.$$

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Ex: $\mathbb{G}_a = (L_\infty(G), \Gamma_a, \varphi_a, \psi_a)$, $\xi \in L_2(G)$, $\|\xi\|_2 = 1$, then $\frac{d\omega'_\xi}{d\varphi} = M_{|\xi|^2}$ and

$$H(|\xi|^2) = -\int_G |\xi(s)|^2 \log|\xi(s)|^2 ds.$$

Theorem (Kunze '58; Terp '80)

Let G be a locally compact group, and let $\xi \in L_2(G)$. The ***Fourier Transform*** *of ξ is the closed densely defined operator on $L_2(G)$ given by*

$$\mathcal{F}(\xi)\eta = \xi * \Delta^{1/2}\eta,$$

where $\mathcal{D}(\mathcal{F}(\xi)) = \{\eta \in L_2(G) \mid \xi * \Delta^{1/2}\eta \in L_2(G)\}$. Moreover, $\mathcal{F} : L_2(G) \to L_2(VN(G), \varphi'_s)$ is an isometric isomorphism.

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Ex: $\mathbb{G}_s = (VN(G), \Gamma_s, \varphi_s)$, $\xi \in C_c(G)$, $\|\xi\|_2 = 1$. If φ_s' is the Plancherel weight on $VN(G)'$, then one can show that

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Thus, $H(\omega_\xi) = H'(\omega'_{J\xi}) = -\left\langle \log\left(|\mathcal{F}(\xi)|^2\right) J\xi, J\xi \right\rangle$.

Uncertainty Principle for Locally Compact Groups

Theorem (C–Kalantar '14)

Let G be a locally compact group and let $\xi \in L^2(G)$ with $\|\xi\|_2 = 1$. If $H(\omega_\xi)$ and $\hat{H}(\omega_{\mathcal{F}\xi})$ are finite, then

$$H(\omega_\xi) + \hat{H}(\omega_{\mathcal{F}\xi}) \geq -\log\|\Delta^{-1/2}\xi\|_2^2,$$

where for $\xi \notin \mathcal{D}(\Delta^{-1/2})$ we let $\|\Delta^{-1/2}\xi\|_2 = \infty$.

Thank you!