Haagerup Approximation Property and positive cones associated with a von Neumann algebra

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May 29, 2014

Workshop on Operator Spaces, Locally Compact Quantum Groups and Amenability

Fields Institute

科研費 KAKENHI

Supported by JSPS KAKENHI Grant Number 25800065.

Definition (Haagerup 1979)

A locally compact group G has the **HAP** if
$\exists$ positive definite functions φ_n on G such that

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Definition (Choda 1983)

A finite v.N. algebra M wiht a f.n. tracial state τ has the **HAP** if
$\exists$ c.c.p. normal maps Φ_n on M such that

(A) $\Phi_n \to \mathrm{id}_M$ in σ-WOT;
(B) $\tau \circ \Phi_n \leq \tau$ and $T_n \in \mathbb{K}(H_\tau)$ satisfying

$$T_n(x\xi_\tau) = \Phi_n(x)\xi_\tau \quad \text{for } x \in M.$$

Standard form

Theorem (Haagerup 1975)

Any v.N. algebra is $*$-isomorphic to a v.N. algebra M on a Hilbert space H such that there exists a conjugate-linear isometric involution J on H and a self-dual positive cone P in H with the following properties:

(1) $JMJ = M'$;

(2) $J\xi = \xi$ for any $\xi \in P$;

(3) $xJxJP \subset P$ for any $x \in M$;

(4) $JcJ = c^*$ for any $c \in Z(M) := M \cap M'$.

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Theorem (Ando-Haagerup 2012)

The condition (4) can be dropped.

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- $\mathcal{A}_\varphi := \Lambda_\varphi(n_\varphi \cap n_\varphi^*)$ is the associated left Hilbert algebra with

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- S_φ is the closure of the conjugate-linear operator $\xi \mapsto \xi^\sharp$ on H_φ, which has the polar decomposition

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Then the quadruple $(\pi_\varphi(M), H_\varphi, J_\varphi, P_\varphi)$ is a standard form.

Let (M, H, J, P) and $(\mathbb{M}_n, \mathbb{M}_n, J_{\mathrm{tr}_n}, \mathbb{M}_n^+)$ be standard forms.

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Definition

- $[\xi_{ij}] \in \mathbb{M}_n(H)$ is ***n*-positive** if

$$\sum_{i,j=1}^{n} x_i J x_j J \xi_{ij} \in P \quad \text{for any } x_1, \ldots, x_n \in M.$$

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Theorem (Schmitt-Wittstock 1982, Miura-Tomiyama 1984)

$(\mathbb{M}_n(M), \mathbb{M}_n(H), J \otimes J_{\mathrm{tr}_n}, P^{(n)})$ is a standard form.

Definition

Let (M, H, J, P) be a standard form.
A bounded linear operator $T\colon H \to H$ is **completely positive** (**c.p.**) if

$$(T \otimes \mathrm{id}_n)P^{(n)} \subset P^{(n)} \quad \text{for any } n \geq 1.$$

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A v.N. algebra M has the **HAP** if
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Remark

The HAP does not depend on the choice of (M, H, J, P).

Theorem (Torpe 1981, Junge-Ruan-Xu 2005)

A v.N. algebra M is injective
$\iff \exists$ finite rank c.c.p. T_n on H such that $T_n \to 1_H$ in SOT.

Permanece property

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- If $p_n \in M$ are projections with $p_n \nearrow 1_M$,
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- M has the HAP $\Longleftrightarrow$ M' has the HAP.

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Corollary (O-Tomatsu 2013)

A v.N. algebra M has the HAP if and only if so does its core $\widetilde{M} := M \rtimes_\sigma \mathbb{R}$.

Conditional expactation

Theorem (O-Tomatsu 2013)

If $E\colon M \to N$ is a (**not necessarily normal**) conditional expectation and M has the HAP, then N has the HAP.

σ-finite v.N. algebras

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Remark

Our HAP is equivalent to the original definition when M is finite.

Definition (Caspers-Skalski 2013)

A (σ-finite) v.N. algebra M (with a f.n. state φ) has the **CS-HAP** if

- $\exists$ compact contractions T_n on H_φ such that $T_n \to 1_{H_\varphi}$ in SOT;
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Remark

The OT-HAP is equivalent to the CS-HAP.

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M has the HAP if and only if its core $\widetilde{M} := M \rtimes_\sigma \mathbb{R}$ has the HAP.

Equivalence between CS-HAP and OT-HAP

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In the case of finite v.N. algebras, CS-HAP and OT-HAP are equivalent to the original one.

CS-HAP and OT-HAP

Let M be a σ-finite v.N. algebra M with a f.n. state φ.

OT-HAP

- $\exists$ c.p. compact contractions T_n on H_φ such that $T_n \to 1_{H_\varphi}$ in SOT;
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- $J_\varphi P_\varphi^\alpha = P_\varphi^{1/2-\alpha}$;
- $P_\varphi^{1/2-\alpha} = \{\eta \in H_\varphi : \langle \eta, \xi \rangle \geq 0 \text{ for } \xi \in P_\varphi^\alpha\}$.

Let $0 \leq \alpha \leq 1/2$. Let M be a v.N. algebra with a f.n.s weight φ.

Definition (O-Tomatsu 2014)

A v.N. algebra M has the **α-HAP** if
$\exists$ contractiions $T_n \in \mathbb{K}(H_\varphi)$ such that

- $T_n \to 1_{H_\varphi}$ in SOT;
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Remark

It can be proved that the α-HAP does not depend on the choice of φ.

Let M be a von Neumann algebra.

Theorem (O-Tomatsu 2014)

The following are equivalent:

(1) M has the OT-HAP, i.e., $1/4$-HAP;

(2) M has the CS-HAP;

(3) M has the 0-HAP;

(4) M has the α-HAP for some/all α;

(5) For any f.n.s. weight φ, $\exists$ c.c.p. normal maps Φ_n on M such that
- $\varphi \circ \Phi_n \leq \varphi$;
- $\Phi_n \to \mathrm{id}_M$ in σ-WOT;
- for any $0 \leq \alpha \leq 1/2$, the associated c.c.p. operators T_n^α are compact and $T_n^\alpha \to 1_{H_\varphi}$, where

$$T_n^\alpha(\Delta_\varphi^\alpha \Lambda_\varphi(x)) = \Delta_\varphi^\alpha \Lambda_\varphi(\Phi_n(x)) \quad \text{for } x \in n_\varphi.$$

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Then $T_n^0 = \Delta_\varphi^{-1/4} T_n \Delta_\varphi^{1/4} \in \mathbb{B}(H)$ satisfies $T_n^0(x\xi_\varphi) = \Phi_n(x)\xi_\varphi$,
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We may assume that M is σ-finite with a f.n. state φ.
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Then $T^0_{n,\beta,\gamma}(x\xi_\varphi) := \Phi_{n,\beta,\gamma}(x)\xi_\varphi$ such that

$$T^0_{n,\beta,\gamma} = (U_\beta\Delta_\varphi^{-1/4})T_n(\Delta_\varphi^{1/4}U_\gamma) \in \mathbb{K}(H_\varphi),$$

because $U_\beta\Delta_\varphi^{-1/4}, \Delta_\varphi^{1/4}U_\gamma \in \mathbb{B}(H_\varphi)$.

Independency on the choice of positive cones

Let M be a von Neumann algebra.

Theorem (O-Tomatsu 2014)

The following are equivalent:

(1) M has the OT-HAP, i.e., $1/4$-HAP;

(2) M has the CS-HAP;

(3) M has the 0-HAP;

(4) M has the α-HAP for some/all α;

(5) For any f.n.s. weight φ, $\exists$ c.c.p. normal maps Φ_n on M such that
  - $\varphi \circ \Phi_n \leq \varphi$;
  - $\Phi_n \to \mathrm{id}_M$ in σ-WOT;
  - for any $0 \leq \alpha \leq 1/2$, the associated c.c.p. operators T_n^α are compact and $T_n^\alpha \to 1_{H_\varphi}$, where

$$T_n^\alpha(\Delta_\varphi^\alpha \Lambda_\varphi(x)) = \Delta_\varphi^\alpha \Lambda_\varphi(\Phi_n(x)) \quad \text{for } x \in n_\varphi.$$

Proof of (3) $\Longrightarrow$ (4)

Lemma (O-Tomatsu 2014)

Let $\alpha \in [0, 1/4]$ and $T \in \mathbb{B}(H_\varphi)$ be completely positive with respect to P_φ^α. Then for $\beta \in [\alpha, 1/2 - \alpha]$,

- $\Delta_\varphi^{\beta-\alpha} T \Delta_\varphi^{\alpha-\beta}$ can be extended to a bounded operator on H_φ with the norm $\|T\|$, which is completely positive with respect to P_φ^β.
- If T is compact, then so does $\Delta_\varphi^{\beta-\alpha} T \Delta_\varphi^{\alpha-\beta}$.

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Apply the three lines Theorem.

Let $1 < p < \infty$ and $L^p(M)$ be Haagerup's non-commutative L^p-space.

HAP for non-commutative L^p-spaces

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A v.N. algebra M has the **L^p-HAP** if
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Theorem (O-Tomatsu 2014)

A v.N. algebra M has the HAP, i.e., L^2-HAP
$\iff$ M has the L^p-HAP for some/all $1 < p < \infty$.

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