Product type actions of compact quantum groups

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Product type actions I

Compact quantum group

Definition (Woronowicz)

A *compact quantum group* G is a pair of $C(G)$ and δ s.t.

- $C(G)$: unital C*-algebra.
- $\delta\colon C(G) \to C(G) \otimes C(G)$: coproduct, i.e.

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta.$$

- (Cancellation property) $\delta(C(G)) \cdot (\mathbb{C} \otimes C(G))$ and $\delta(C(G)) \cdot (C(G) \otimes \mathbb{C})$ are dense in $C(G)$.

Notation

We need

- h: the Haar state.
- $L^2(G)$: the GNS Hilbert space.
- $L^\infty(G)$: the weak closure of $C(G)$.

A unitary $v \in B(H) \otimes L^\infty(G)$ is a *representation* if

$$(\mathrm{id} \otimes \delta)(v) = v_{12} v_{13}.$$

Let

- G: a compact quantum group.
- $v \in B(H) \otimes L^\infty(G)$: a unitary representation on H.
- $\gamma\colon B(H) \to B(H) \otimes L^\infty(G)$ defined by

$$\gamma(x) = v(x \otimes 1)v^* \quad \text{for } x \in B(H).$$

$\rightsquigarrow$ γ is an action, that is,

$$(\gamma \otimes \mathrm{id}) \circ \gamma = (\mathrm{id} \otimes \delta) \circ \gamma.$$

Assumption (not essential): γ is faithful.
Namely, any irreducible representation of G is contained in $(v \otimes \overline{v})^{\otimes n}$ for a large n.

Product type actions

If G: a compact group,

$\rightsquigarrow$ a product type action Ad $v^{\otimes\infty}$ is minimal, i.e. $(\mathcal{M}^{\alpha})' \cap \mathcal{M} = \mathbb{C}$.

Let $v^{\otimes n}$, tensor product representations.

Then the actions Ad $v^{\otimes n}$ extend to the following UHF-algebra:

$$B(H) \to B(H)^{\otimes 2} \to \cdots \to B(H)^{\otimes n} \to \cdots \to B(H)^{\otimes\infty}.$$

Fix an invariant state ϕ on $B(H)$ for Ad v:

$$(\phi \otimes \mathrm{id})(v(x \otimes 1)v^*) = \phi(x)1, \quad \forall x \in B(H).$$

Denote by $\mathcal{M}$ the weak closure w.r.t. the product state φ:

$$(\mathcal{M}, \varphi) := \bigotimes_{n=1}^{\infty} (B(H), \phi)''.$$

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Then set the product type action $\alpha := \operatorname{Ad} v^{\otimes\infty}$ on $\mathcal{M}$.
Recall the fixed point algebra:

$$\mathcal{M}^\alpha := \{x \in \mathcal{M} \mid \alpha(x) = x \otimes 1\}.$$

Our study relies on he following result.

Theorem (Izumi)

Suppose that G is not of Kac type (h is non-tracial). Then the following statements hold:

- $(\mathcal{M}^\alpha)' \cap \mathcal{M} \neq \mathbb{C}$.
- $(\mathcal{M}^\alpha)' \cap \mathcal{M}$ *is isomorphic to the Poisson boundary* $H^\infty(\widehat{G}, \mu)$, *which is determined by a random walk* μ *on the dual* $\widehat{G}$.

$\rightsquigarrow$ non-minimality of $\alpha = \operatorname{Ad} v^{\otimes\infty}$.

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Quantum flag manifolds

Quick review of the recipe of G_q. Let $0 < q < 1$.

- A Cartan matrix $A = (a_{ij})_{i,j \in I}$ (finite, irreducible).
- The root data $(\mathfrak{h}, \{h_i\}_{i \in I}, \{\alpha_i\}_{i \in I})$.
- Drinfel'd–Jimbo's quantum group $U_q(\mathfrak{g})$.
- Collect $*$-representations $\pi \colon U_q(\mathfrak{g}) \to B(H)$ (admissible ones).
- For $\xi, \eta \in H$, set $C^\pi_{\xi,\eta}(x) := \langle \pi(x)\eta, \xi \rangle$ for $x \in U_q(\mathfrak{g})$.
- $$A(G_q) := \mathrm{span}\{C^\pi_{\xi,\eta} \mid \pi, \xi, \eta\} \subset U_q(\mathfrak{g})^*.$$

  $\rightsquigarrow A(G_q)$ inherits the Hopf $*$-algebra structure from $U_q(\mathfrak{g})^*$.
- $C(G_q) :=$ the universal C*-algebra of $A(G_q)$.

  $\rightsquigarrow C(G_q)$ is a compact quantum group with faithful Haar state.

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Maximal torus, Quantum flag manifold

Let $T := \mathbb{T}^I$, the $|I|$-fold direct product group of $\mathbb{T}$.
$\rightsquigarrow$ T is a closed subgroup of G_q, that is,
$\exists$ a canonical surjective $*$-homomorphism $r_T \colon C(G_q) \to C(T)$ s.t.

$$\delta_T \circ r_T = (r_T \otimes r_T) \circ \delta_{G_q}.$$

$\rightsquigarrow$ We call T the maximal torus of G_q.
The quantum flag manifold is defined by

$$C(T \backslash G_q) := \{x \in C(G_q) \mid (r_T \otimes \mathrm{id})(\delta_{G_q}(x)) = 1 \otimes x\}.$$

Then δ_{G_q} provides $C(T \backslash G_q)$ with a (right) action of G_q.

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Our main ingredients are the following two results.
Recall a product type action $\alpha\colon \mathcal{M} \to \mathcal{M} \otimes L^\infty(G_q)$.

Theorem (Izumi, Izumi-Neshveyev-Tuset, T)

One has the following G_q-equivariant isomorphisms:

$$L^\infty(T\backslash G_q) \cong H^\infty(\widehat{G_q}) \cong (\mathcal{M}^\alpha)' \cap \mathcal{M}.$$

Remark

- The Poisson boundary $H^\infty(\widehat{G_q})$ does not depend on a choice of a generating probability measure μ.
- $Z(\mathcal{M}^\alpha) \cong H^\infty(\ell^\infty(\mathrm{Irr}(G_q))) = \mathbb{C}$ (Hayashi).
  $\rightsquigarrow \mathcal{M}^\alpha$ is a factor.
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The second one is about the structure of $L^\infty(G_q)$.

Theorem (T)

The following statements hold:

- $L^\infty(T\backslash G_q)$ *is a factor of type* I_∞.
- $L^\infty(T\backslash G_q)' \cap L^\infty(G_q) = Z(L^\infty(G_q))$.
  Thus $L^\infty(G_q) = Z(L^\infty(G_q)) \vee L^\infty(T\backslash G_q)$.
- *The left action* γ *of* T *on* $Z(L^\infty(G_q))$ *is faithful and ergodic.*

Proof.

Let $\Theta\colon L^\infty(T\backslash G_q) \to H^\infty(\widehat{G_q})$ be the Poisson integral ($\widehat{G_q}$-G_q-isomorphism).

Then Θ maps $Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q)$ into $L^\infty(\widehat{G_q})^{\widehat{G_q}} = \mathbb{C}$.

$\rightsquigarrow$

$$Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q) = \mathbb{C}.$$

$\rightsquigarrow \gamma\colon T \curvearrowright Z(L^\infty(G_q))$ is ergodic.

Let $C^\lambda_{\lambda,w_0\lambda} = v|C^\lambda_{\lambda,w_0\lambda}|$ be the polar decomposition.

$\rightsquigarrow$ v is central.

$\rightsquigarrow$ γ is faithful on the center.

$\rightsquigarrow$ $L^\infty(G_q) = Z(L^\infty(G_q)) \vee L^\infty(T\backslash G_q)$.

It is well-known that $L^\infty(G_q)$ is of type I. □

Product type actions II

Tensor product decomposition

Recall

- $\alpha = \mathrm{Ad}\, v^{\otimes\infty} \colon \mathcal{M} \to \mathcal{M} \otimes L^\infty(G)$.
- $\mathcal{Q} := (\mathcal{M}^\alpha)' \cap \mathcal{M} \cong L^\infty(T \backslash G_q) \cong B(\ell^2)$.

Therefore, we have a tensor product decomposition,

$$\mathcal{M} = \mathcal{R} \vee \mathcal{Q} \cong \mathcal{R} \otimes \mathcal{Q},$$

where $\mathcal{R} := \mathcal{Q}' \cap \mathcal{M} = ((\mathcal{M}^\alpha)' \cap \mathcal{M})' \cap \mathcal{M}$.
Then

- $\mathcal{M}^\alpha \subset \mathcal{R}$ is irreducible, i.e. $(\mathcal{M}^\alpha)' \cap \mathcal{R} = \mathbb{C}$
- $\mathcal{M}^\alpha \subset \mathcal{R}$ is of depth 2.

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So, $\exists$ a minimal action $\beta\colon H \curvearrowright \mathcal{R}$ s.t. $\mathcal{M}^\alpha = \mathcal{R}^\beta$.
What is a compact quantum group H?
The irreducible decomposition of the bimodule ${}_{\mathcal{M}^\alpha}L^2(\mathcal{R})_{\mathcal{M}^\alpha}$ implies the following.

Theorem (T)

The subfactor $\mathcal{M}^\alpha \subset \mathcal{R}$ comes from a minimal action β of the maximal torus T on $\mathcal{R}$.

Namely, $H = T$.
Actually, $\beta_t =$ the restriction of α_t on $\mathcal{R}$ though this fact is non-trivial at first.

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To study β, we need the canonical generators of $Z(L^\infty(G_q))$.

Recall $\gamma\colon T \curvearrowright Z(L^\infty(G_q))$ is faithful and ergodic.

$\rightsquigarrow Z(L^\infty(G_q)) \cong L^\infty(T)$.

$\rightsquigarrow Z(L^\infty(G_q)) = \{v_\lambda \mid \lambda \in \widehat{T}\}''$, where v_λ is a unitary with

$$v_\lambda v_\mu = v_{\lambda+\mu}, \quad \gamma_t(v_\lambda) = \langle t, \lambda \rangle v_\lambda.$$

Then

$$\begin{aligned} L^\infty(G_q) &= Z(L^\infty(G_q)) \vee L^\infty(T\backslash G_q) \\ &= \{v_\lambda \mid \lambda \in \widehat{T}\}'' \vee L^\infty(T\backslash G_q). \end{aligned}$$

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Assumption: $\mathcal{M}^\alpha$ is infinite.
Then the minimal action $\beta\colon T \curvearrowright \mathcal{R}$ is dual, that is,

$$\mathcal{R} = \mathcal{M}^\alpha \vee \{u_\lambda \mid \lambda \in \widehat{T}\}'' \cong \mathcal{M}^\alpha \rtimes_\theta \widehat{T},$$

where $\theta_\lambda = \operatorname{Ad} u_\lambda$ on $\mathcal{M}^\alpha$, $u_\lambda u_\mu = u_{\lambda+\mu}$.
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$$\mathcal{M} = \mathcal{R} \vee \mathcal{Q} = \mathcal{M}^\alpha \vee \{u_\lambda \mid \lambda \in \widehat{T}\}'' \vee \mathcal{Q}.$$

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How do δ and α act on v_λ and u_λ, respectively?
Set w_λ and w_λ^o as follows:

$$\delta(v_\lambda) = (v_\lambda \otimes 1)w_\lambda, \quad \alpha(u_\lambda) = (u_\lambda \otimes 1)w_\lambda^o$$

Then $w_\lambda, w_\lambda^o \in L^\infty(T\backslash G_q) \otimes L^\infty(G)$ by regarding $\mathcal{Q} = L^\infty(T\backslash G_q)$.
Obviously they are one-cocycles of $\delta\colon L^\infty(T\backslash G_q) \curvearrowleft G_q$, that is,

$$(w \otimes 1)(\delta \otimes \mathrm{id})(w) = (\mathrm{id} \otimes \delta)(w).$$

Moreover, for $x \in L^\infty(T\backslash G_q)$:

$$w_\lambda \delta(x) w_\lambda^* = (v_\lambda^* \otimes 1)\delta(v_\lambda x v_\lambda^*)(v_\lambda \otimes 1) = \delta(x),$$

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Invariant cocycles

Namely, w_λ, w_λ^o belong to the following set:

$$Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q)) \\ := \{w \in L^\infty(T\backslash G_q) \otimes L^\infty(G_q) \mid \delta\text{-cocycle},\ \delta^w = \delta \text{ on } L^\infty(T\backslash G_q)\}.$$

Thus we must determine those invariant cocycles.

Theorem (T)

$$Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q)) = \{w_\lambda \mid \lambda \in \widehat{T}\}.$$

$\rightsquigarrow$ $w_\lambda = w_\lambda^o$ up to an automorphism of $\widehat{T}$.

$\rightsquigarrow$ $\exists$ a G_q-equivariant embedding:

$$L^\infty(G_q) = \{v_\lambda \mid \lambda \in \widehat{T}\}'' \vee L^\infty(T\backslash G_q) \cong \{u_\lambda \mid \lambda \in \widehat{T}\}'' \vee \mathcal{Q} \subset \mathcal{M}.$$

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Using this embedding, we obtain our main result.

Theorem (T)

A faithful product type action of G_q is induced from a minimal action of T on a type III factor. The minimal action is uniquely determined up to conjugacy.

We will give a sketch of a proof of the equality,

$$Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q)) = \{w_\lambda \mid \lambda \in \widehat{T}\},$$

where w_λ is the canonical cocycle, that is,

$$\delta(v_\lambda) = (v_\lambda \otimes 1)w_\lambda, \quad \lambda \in \widehat{T}.$$

Sketch of a proof

- Show that the perturbed action δ^w is ergodic on $L^\infty(G_q)$.
- By 2×2-matrix trick, take a unitary $v \in L^\infty(G_q)$ such that

$$\delta(v) = (v \otimes 1)w.$$

- By Fourier type expansion, we have

$$v = \sum_{\lambda \in \widehat{T}} v_\lambda a_\lambda,$$

where $a_\lambda \in L^\infty(T \backslash G_q)$.
In fact, there exists a unique λ such that $v = v_\lambda a_\lambda$.
We want to show that $a_\lambda \in \mathbb{C}$.

Sketch of a proof

- Show that the perturbed action δ^w is ergodic on $L^\infty(G_q)$.
- By 2×2-matrix trick, take a unitary $v \in L^\infty(G_q)$ such that

$$\delta(v) = (v \otimes 1)w.$$

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We want to show that $a_\lambda \in \mathbb{C}$.

Since $\delta^w = \delta$ on $L^\infty(T\backslash G_q)$, we have the following equality putting $\theta := \operatorname{Ad} a_\lambda$:

$$\delta \circ \theta = (\theta \otimes \mathrm{id}) \circ \delta,$$

which means that θ is a G_q-equivariant automorphism on $L^\infty(T\backslash G_q)$.
The following result shows that a_λ is a scalar.

Theorem

$\mathrm{Aut}_{G_q}(L^\infty(T\backslash G_q)) = \{\mathrm{id}\}$.

This follows from the following result:

Theorem (Dijkhuizen-Stokman)

The counit is the unique character of $C(T\backslash G_q)$.

Indeed, we have $\varepsilon \circ \theta = \varepsilon$ on $C(T\backslash G_q)$, and

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Classification

$SU_q(2)$ case

Let $G_q = SU_q(2)$.
$\rightsquigarrow$ T is the one-dimensional torus.

Aim: Classification of product type actions up to cocycle conjugacy.

Recall $\mathcal{M} = \mathcal{R} \vee \mathcal{Q}$, $\mathcal{Q} = (\mathcal{M}^\alpha)' \cap \mathcal{M}$ and $\beta \colon T \curvearrowright \mathcal{R}$.
It is not hard to show the following.

Lemma

The minimal action β_t on $\mathcal{R}$ is cocycle conjugate to α_t on $\mathcal{M}$.

$\rightsquigarrow$ β is (invariantly) approximately inner,

$\rightsquigarrow$ $\hat{\beta} \colon \mathbb{Z} \curvearrowright \mathcal{R} \rtimes_\beta T$ is centrally free.

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Classification results

It depends on a type of $\mathcal{M}^\alpha$.

Theorem

If $\mathcal{M}^\alpha$ is of type II, then α is unique up to conjugacy. Indeed, α is conjugate to $\operatorname{Ind}_T^{G_q} \sigma^{\varphi_q}_{t/\log q}$, where φ_q denotes the Powers state of type III_q.

In particular, $\mathcal{M}^\alpha$ and $\mathcal{M}$ must be of type II_1 and III_q.

Corollary

For $0 < \lambda < 1$ with $\lambda \neq q$, $\operatorname{Ind}_T^{G_q} \sigma^{\varphi_\lambda}_{t/\log \lambda}$ is mutually non-conjugate and non-product type actions of $SU_q(2)$.

Theorem

If $\mathcal{M}^\alpha$ is of type III_1, then α is unique up to conjugacy. Indeed, α is conjugate to $\operatorname{Ind}_T^{G_q}(\mathrm{id}_{\mathcal{R}_\infty} \otimes m)$, where m denotes the unique minimal action of T on $\mathcal{R}_0$.

In fact, this result holds for a general G_q.

Proof.

May assume that $\mathcal{R} = \mathcal{M}^\alpha \rtimes_\theta \widehat{T}$.

$\beta = \hat{\theta}$ is invariantly approximately inner

$\rightsquigarrow$ θ has the Rohlin property $\rightsquigarrow$ θ is centrally free.
& $\mathrm{Aut}(\mathcal{M}^\alpha) = \overline{\mathrm{Int}}(\mathcal{M}^\alpha)$ (Kawahigashi–Sutherland–Takesaki).
Thus θ is cocycle conjugate to $\mathrm{id}_{\mathcal{R}_\infty} \otimes \theta^0$ (Ocneanu),
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When $\mathcal{M}^\alpha$ is of type III_λ, write $\mathcal{R} = \mathcal{M}^\alpha \rtimes_\theta \mathbb{Z}$.
We know θ^n is not centrally trivial ($=$ not modular).
So, the automorphism θ is classified by Connes–Takesaki module $\mathrm{mod}(\theta) \in \mathbb{R}_{>0}/\lambda^{\mathbb{Z}} = [\lambda, 1)$.

Theorem

Let $0 < \lambda < 1$. If $\mathcal{M}^\alpha$ is of type III_λ, then $\mathrm{mod}(\theta) = q$ or $\lambda^{1/2}q$ in $\mathbb{R}_{>0}/\lambda^{\mathbb{Z}}$. In each case, α is unique up to conjugacy.

This immediately implies the following result.

Corollary

Let $0 < \lambda < 1$.
Suppose that $\mu \in \mathbb{R}$ satisfies $0 < \mu < 1$ and $\mu \notin (\lambda^{1/2})^{\mathbb{Z}_+}$.
Then $\mathrm{Ind}_T^{G_q}(\mathrm{id}_{\mathcal{R}_\lambda} \otimes \sigma^{\varphi_\mu}_{t/\log \mu})$ is not of product type.
In particular, for any such λ, there exist uncountably many, non-product type, mutually non-cocycle conjugate actions of $SU_q(2)$ on $\mathcal{R}_\infty$ with type III_λ fixed point factor.

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Related problem

We know that $L^\infty(T\backslash G_q)$ is a type I factor.
Actually, the right action δ is implemented by a unitary:

$$\delta(x) = U(x \otimes 1)U^*, \quad x \in L^\infty(T\backslash G_q).$$

Then the following Ω satisfies the 2-cocycle relation:

$$U_{12}U_{13} = (\mathrm{id} \otimes \delta)(U)(1 \otimes \Omega^*).$$

Then the twisted bialgebra $G_{q,\Omega} = (L^\infty(G_q), \delta_\Omega)$ is again a (locally compact) quantum group (De Commer).

Problem

Realize $G_{q,\Omega}$ as a concrete quantum group.

If $G_q = SU_q(2)$, then $G_{q,\Omega} \cong \widetilde{E}_q(2)$ (De Commer).

Thank you!