Coamenability and quantum groupoids (work in progress)

Leonid Vainerman

University of Caen

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Contents

Introduction : Coamenable compact quantum groups

Theorem and Definition [E.Bédos,G.J.Murphy,L.Tuset]

A compact quantum group $\mathbb{G} = (A, \Delta)$ (A is a (separable) unital C^* - algebra, $\Delta : A \to A \otimes A$) is called coamenable if one of the following equivalent conditions holds :

- The counit ε extends continuously to $A_{red} := \pi_h(A)$ (π_h comes from the Haar state h).
- The C^* - algebra A is isomorphic to A_{red}.
- h is faithful and ε is bounded with respect to $|| \cdot ||_A$.
- There is a non-zero $\star$-homomorphism $\pi : A_{red} \to \mathbb{C}$.

Examples

"Trivial" examples

1. A countable discrete group Γ is called amenable iff $C^*(\Gamma) \cong$
$\cong C^*_{red}(\Gamma)$. So the compact quantum group $(C^*(\Gamma), \Delta)$ (where
$\Delta : \lambda_\gamma \mapsto \lambda_\gamma \times \lambda_\gamma$) is coamenable iff Γ is amenable.

2. If G is a Hausdorff compact group, then $(C(G), \Delta)$ (where
$(\Delta f)(g,h) = f(gh)$) is coamenable. Indeed, the counit
$\varepsilon : f(g) \mapsto f(e)$ is bounded

Example [T.Banica]

The compact quantum group $C(SU_q(2))$ $(q > 0)$ is coamenable.

One of the proofs uses the notions of a fusion ring and a fusion algebra of corepresentations of a compact quantum group.

Fusion algebras

Definition [F.Hiai,M.Izumi]

A fusion algebra is a unital algebra R with a basis I over $\mathbb{Z}$ s. t. :

- $$\zeta\eta = \Sigma_\alpha \; N_{\zeta,\eta}^{\alpha}\alpha \qquad \forall \zeta, \eta \in I,$$

where $N_{\zeta,\eta}^{\alpha} \in \mathbb{Z}^+$, only finitely many nonzero.

- There is a bijection $\zeta \mapsto \overline{\zeta}$ of I which extends to a $\mathbb{Z}$-linear anti-multiplicative involution of R.

- Frobenius reciprocity :

$$N_{\zeta,\eta}^{\alpha} = N_{\overline{\zeta},\alpha}^{\eta} = N_{\alpha,\overline{\eta}}^{\zeta} \qquad \forall \zeta, \eta, \alpha \in I.$$

- There is a dimension function $d : I \to [1, \infty[$ such that $d(\zeta) = d(\overline{\zeta})$ which extends to a $\mathbb{Z}$-linear multiplicative map $R \to \mathbb{R}$.

Examples 1) A group algebra $\mathbb{Z}\Gamma$ of Γ.

2) $R(G)$ of unitary representations of G.

3) $R(\mathbb{G})$ of unitary corepresentations of $\mathbb{G}$.

Definition A fusion algebra R is called amenable if $1 \in \sigma(\lambda_\mu)$ (*)

for any finitely supported, symmetric probability measure μ on I,

$$\text{where} \quad \lambda_\mu := \Sigma_{\zeta \in I} \mu(\zeta) \lambda_\zeta, \ \lambda_\zeta(f)(\eta) := \Sigma_{\alpha \in I} f(\alpha) \frac{d(\alpha)}{d(\zeta) d(\eta)} N^\alpha_{\zeta,\eta}$$

is a left translation operator in $l^2(I, d^2)$.

Remark In case 1) (*) is equivalent to the existence of an invariant mean on Γ but in general (*) is strictly stronger (see [HI]).

Theorem [F.Hiai,M.Izumi],[D.Kyed] A compact quantum group $\mathbb{G}$ is coamenable if and only if $R(\mathbb{G})$ is amenable.

Corollary $C(SU_q(2))$ is coamenable because $R(C(SU_q(2))) \cong$

$\cong R(SU(2))$ which is known to be amenable.

Locally compact groupoids

A groupoid is a small category with all morphisms invertible.

G is the set of morphisms, G^0 is the set of objects, the source and the range maps $s, r : G \to G^0$; the inverse $\gamma \mapsto \gamma^{-1}$ is such that $s(\gamma) = r(\gamma^{-1}),\ r(\gamma) = s(\gamma^{-1})$;

the composition (multiplication) $G_s \times_r G := \{(\alpha, \beta) \in G \times G | s(\alpha) = r(\beta)\} \to G$ is associative.

Topology : G is Hausdorff, second countable l.c., G^0 is compact, s, r are surjective, open, continuous. G is called étale if s and r are local homeomorphisms.

A continuous left Haar system on G : a family $\lambda = (\lambda^x)_{x \in G^0}$ of positive Radon measures such that $supp(\lambda^x) = G^x := r^{-1}(x)$, $\gamma \lambda^{s(\gamma)} = \lambda^{r(\gamma)}$, $x \mapsto \int_{G^x} f d\lambda^x$ is continuous $(\gamma \in G, f \in C_c(G))$.
Then $\lambda^{-1} := (\lambda^x(\gamma^{-1}))_{x \in G^0}$ is a right Haar system.

Groupoid C^*-algebras

We call (G, λ, μ) a measured groupoid if a probability measure μ is quasi-invariant : $supp(\mu) = G^0$ and $\nu \cong \nu^{-1}$, where $\nu = \mu \circ \lambda$. Then $I(G) := \{f |\ ||f||_I < \infty\}$, where $||f||_I =$

$$= max\{|| \int_{G^x} |f(\gamma)| d\lambda^x(\gamma)||_\infty, || \int_{G^x} |f(\gamma)| d\lambda^x(\gamma^{-1})||_\infty\}$$

is a Banach $\star$-algebra with

$$(f \star g)(\gamma') = \int_{G^{\gamma'}} f((\gamma')^{-1}\gamma) g(\gamma) d\lambda^{r(\gamma')}(\gamma), \quad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

having a two-sided approximate identity. The construction of full $C^*(G, \lambda, \mu)$ is standard. Left regular representation on $L^2(G, \nu)$:

$$L(f)g(\gamma') := \int_{G_{\gamma'}} f(\gamma) D^{-1/2}(\gamma) g(\gamma^{-1}\gamma') d\lambda^{r(\gamma')}(\gamma) \quad (D := \frac{d\nu}{d\nu^{-1}}),$$

then $C^*_{red}(G, \lambda, \mu)$ is $\overline{C_c(G)}$ with respect to $||f||_{red} := ||L(f)||$.

Amenable groupoids [C.Anantharaman-Delaroche and J.Renault]

Definition We say that a measured groupoid (G, λ, μ) is amenable if there is an invariant mean, i.e., a positive unital $L^\infty(G^0, \mu)$-linear map $m : L^\infty(G, \nu) \to L^\infty(G^0, \mu)$ such that $f \star m = (\lambda(f) \circ r)m$ ($f \star m^u (u \in G^0)$ is defined by bitransposition for any $f \in C_c(G)$).

Theorem (i) (G, λ, μ) is amenable iff the trivial representation

$$\varepsilon : f \mapsto \int_{G^x} f(\gamma) D^{-1/2}(\gamma) d\lambda^x(\gamma)$$

of $C^*(G, \lambda, \mu)$ acting on $L^2(G^0, \mu)$, is weakly contained in the regular one.

(ii) If (G, λ, μ) is amenable, then $C^*(G, \lambda, \mu) = C^*_{red}(G, \lambda, \mu)$.

Remark The converse statement to (ii) is not known.

Hopf $\star$-algebroid over commutative base [J.-H. Lu]

$\mathbb{G} = (A, B, r, s, \Delta, \varepsilon, S)$, where A and $B = B^{op}$ are unital $\star$-algebras; $s, r : B \to A$ are unital embeddings, $[s(B), r(B)] = 0$.

So ${}_rA_s$ and $A \otimes_B A := A \otimes A/\{as(b) \otimes a' - a \otimes r(b)a' | a, a' \in A, b \in B\})$ are B-bimodules and unital $\star$-algebras.

Coproduct $\Delta : A \to A \otimes_B A$, counit $\varepsilon : A \to B$ and antipode $S :_r A_s \to_s A_r$ are B-bimodule and $\star$-algebra maps such that :

$$\Delta(s(b)r(c)) = r(c) \otimes_B s(b) \text{ for all} b, c \in B,$$

$(id \otimes_B \Delta) \circ \Delta = (\Delta \otimes_B id) \circ \Delta, \quad (id \otimes_B \varepsilon) \circ \Delta = (\varepsilon \otimes_B id) \circ \Delta = id,$
$S(r(b)) = s(b)$, $S(a_{(1)})a_{(2)} = s(\varepsilon(a))$, $a_{(1)}S(a_{(2)}) = r(\varepsilon(a))$ for all $a \in A, b \in B$, and $\Delta \circ S = \Sigma(S \otimes_B S)\Delta$ (Σ is a "flip").

C^*-algebraic Compact Quantum Groupoid [T.Timmermann]

$\mathbb{G} = (B, \mu, A, r, s, \psi, \Delta, R)$, where $A, B = B^{op}$ are unital C^*-algebras, $r, s : B \to A$ are unital C^*-embeddings, $[s(B), r(B)] = 0$, $R : A \to A$ an involutive C^*-anti-automorphism s.t. $R \circ r = s$, μ is a faithful trace on B, $\psi : A \to$ $\to B$ is a completely positive contraction satisfying :

- $s \circ \psi : A \to s(B)$ is a unital conditional expectation
- $\nu = \mu \circ \psi \circ R$ and $\nu^{-1} = \mu \circ \psi$ are KMS states on A
- $\Delta : A \to A \odot_B A$ a C^*-morphism such that

$$(id \odot_B \Delta) \circ \Delta = (\Delta \odot_B id) \circ \Delta, \quad \Delta \circ R = \Sigma(R \odot_B R)\Delta,$$

($A \odot_B A$ is a minimal fiber C^*-product over B, extending $\otimes_{min}$).

- ψ is strongly invariant : $(\psi \odot_B id)\Delta(a) = s(\psi(a))$ and

$$R[(\psi \odot_B id)(d \odot_B 1)\Delta(a)] = (\psi \odot_B id)(a \odot_B 1)\Delta(d), \ \forall a, d \in A$$

Terminology $(B, \mu, A, r, s, \Delta)$ is called a Hopf C^*-bimodule

C^*-pseudo-multiplicative unitary [T.Timmermann]

The relative tensor product $H \otimes_B K$ of Hilbert C^*-modules over unital $B = B^{op}$ is parallel to the Connes' one. A C^*-pseudo-multiplicative unitary : $V : H \otimes_B H \to H \otimes_B H$ s.t. $V_{12}V_{13}V_{23} = V_{23}V_{12}$. Baaj-Skandalis's approach allows to get Banach algebras

$$A_0 := \{(\omega \otimes_B id)(V) | \omega \in L(H)_*\}, \ \widehat{A_0} := \{(id \otimes_B \omega)(V) | \omega \in L(H)_*\},$$

then Hopf C^*-bimodules $A_{red} = \overline{A_0}$ and $\widehat{A_{red}} = \overline{\widehat{A_0}}$ with coproducts $\Delta : A_{red} \to M(A_{red} \odot_B A_{red})$ and $\hat{\Delta} : \widehat{A_{red}} \to M(\widehat{A_{red}} \odot_B \widehat{A_{red}})$.

Example 1. If $(G, G^0, r, s, \lambda, \mu)$ is a l.c. measured groupoid, put

$$Vf(x, y) := f(x, x^{-1}y), \ \forall f \in C_c(G_r \times_r G).$$

$A_{red} = C^*_{red}(G)$, $\widehat{A_{red}} = C_0(G)$, $\Delta(L(x)) = L(x) \odot_B L(x)$, where $L(x)g(y) := g(x^{-1}y)$ if $x \in G^y$ and 0 otherwise, $g \in C_c(G)$, $M(\widehat{A_{red}} \odot_B \widehat{A_{red}}) = C_b(G_s \times_r G)$, $\hat{\Delta}(f)(x, y) = f(xy)$.

Reduced Hopf C^*-bimodule of a Compact Quantum Groupoid

Example 2. Given $\mathbb{G} = (B, \mu, A, r, s, \psi, \Delta, R)$,

let $H := L^2(A, \nu)$ and $H \otimes_B H$ be the relative tensor product.

Define the fundamental unitary $V : H \otimes_B H \to H \otimes_B H$ by

$$V(a \odot_B a') := [(R \odot_B id)\Delta(a')](a \odot_B 1),$$

Then $A_{red} = \pi_\nu(A)$ is the reduced Hopf C^*-bimodule of $\mathbb{G}$.

Using the theory of fixed and cofixed vectors of pseudo-multiplicative unitaries extending the one of Baaj-Skandalis, one shows that A_{red} is equipped with a bounded right Haar weight and $\widehat{A_{red}}$ - with a bounded counit.

Coamenable Compact Quantum Groupoids

Definition We call a compact quantum groupoid $\mathbb{G}$ coamenable if its reduced C^*-Hopf bimodule has a bounded counit.

Proposition (i) $\mathbb{G}$ is coamenable if and only if its Haar integrals are faithful and it has a bounded counit.

(ii) If $\mathbb{G}$ is coamenable, then A and A_{red} are isomorphic.

Corollary (i) Tensor product of two compact quantum groupoids is coamenable if and only if both of them are coamenable.

(ii) If $\mathbb{G}$ is coamenable, then $\mathbb{G} = \mathbb{G}_{univ}$ (the construction of $\mathbb{G}_{univ}$ can be done using representations and corepresentations of the fundamental unitary of $\mathbb{G}$ along the lines of Baaj-Skandalis).

Remark Unfortunately, there is no "Peter-Weyl type" theory for compact quantum groupoids available at this moment.

Example 1 : continuous functions on a compact groupoid

Let $(G, G^0, r, s, \lambda, \mu)$ be a compact measured groupoid.

Put $A := C(G)$, $B := C(G^0)$,

$[r(h)](\gamma) := h(r(\gamma))$, $[s(h)](\gamma) := h(s(\gamma))$,

$\mu(h) := \int_{G^0} h(x) d\mu(x)$, $Rf(\gamma) := f(\gamma^{-1})$,

$\psi \circ R(f) := \int_{G^x} f(\gamma) d\lambda^x(\gamma)$, $\psi(f) := \int_{G_x} f(\gamma) d\lambda^x(\gamma^{-1})$,

where $h \in C(G^0)$, $f \in C(G), G_x = s^{-1}(x)$.

Finally, identify $A \odot A$ with $C(G_s \times_r G)$ and define,

for any $f \in C(G)$ and $(x, y) \in G_s \times_r G$, $\Delta(f) := f(xy)$.

Then we have an abelian C^*-algebraic compact quantum groupoid

with a bounded counit $\varepsilon : A \to B$, namely $\varepsilon : f \to f|_{G_0}$. Also,

$A_{red} = A = C(G)$.

Example 2 : C^*-algebra of an étale r-discrete groupoid

Let $(G, G^0, r, s, \lambda, \mu)$ be an étale r-discrete measured groupoid (i.e., G^x are countable and λ^x are counting measures, $\forall x \in G^0$).

Put $A := C^*_{red}(G)$ with unit $\mathbf{1_{G^0}}$, $B := C(G^0)$, $r(h) = s(h) :=$ $:= L(h)$, where $h \in C(G^0)$ and $L(f)g := f \star g$ for all $f, g \in C_c(G)$.

Also $\mu(h) =: \int_{G^0} h(x) d\mu(x)$, $R(L(f)) := L(f^+)$, where $f^+(\gamma) :=$ $:= f(\gamma^{-1})$, $\psi(L(f))(x) := f(x^{-1})$.

Finally, $\Delta(L(x)) := L(x) \odot L(x)$, where $L(x)f(y) := f(x^{-1}y)$ if $x \in G^y$ and 0 otherwise, for any $f \in C_c(G)$ and $x, y \in G$.

Then we have a co-commutative C^*-algebraic compact quantum groupoid. G is amenable if and only if the map $\varepsilon : f \mapsto$ $\mapsto \int_{G^x} f(\gamma) D^{-1/2}(\gamma) d\lambda^x(\gamma)$ defines a bounded counit on $C^*_{red}(G)$.

Finite dimensional case : C^*-Weak Hopf algebra [G.Bohm, F.Nill, K.Szlachanyi]

Definition. This is a finite dimensional C^*-bialgebra (A, Δ, ε)

(but $\Delta(1) \neq 1 \otimes 1$ and $\varepsilon(ab) \neq \varepsilon(a)\varepsilon(b)$, in general !) such that

- $(\Delta \otimes \mathrm{id})\Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1))$,
- $\varepsilon(abc) = \varepsilon(ab_{(1)})\varepsilon(b_{(2)}c) = \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c)$, $\forall a, b, c \in A$,

(here $\Delta(b) = b_{(1)} \otimes b_{(2)}$ - Sweedler notation)

- Antipode $S : A \longrightarrow A$ is a bialgebra anti-isomorphism such that

$$m(\mathrm{id} \otimes S)\Delta(a) = \varepsilon(1_{(1)}a)1_{(2)}, \; m(S \otimes \mathrm{id})\Delta(a) = 1_{(1)}\varepsilon(a1_{(2)}),$$

$$S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a).$$

Tensor product is usual !

Nice features

- Dual vector space is again a weak C^*-Hopf algebra
- A C*-quantum groupoid is a quantum group (G.I. Kac algebra)

if and only if either $\Delta(1) = 1 \otimes 1$ or $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$.

- Bases : the C*-subalgebras $B_r := Im(\varepsilon_r)$ and $B_s : Im(\varepsilon_s)$, where

$$\varepsilon_r(a) = m(\mathrm{id} \otimes S)\Delta(a), \quad \varepsilon_s(a) = m(S \otimes \mathrm{id})\Delta(a), \ \forall a \in A.$$

- Reconstruction theorem (T. Hayashi) :

Any fusion category (i.e., tensor and finite semi-simple) is

equivalent to the category of representations of some canonical

weak Hopf algebra with commutative bases.

This gives many non-trivial examples of weak C^*-Hopf algebras.

- II_1-subfactors of finite index and finite depth can be completely characterized in terms of weak C^*-Hopf algebras [D.Nikshych,L.V.]

Example : Temperley-Lieb algebras

Generators : $e_i^2 = e_i = e_i^*$

Relations :

$$e_i e_{i\pm 1} e_i = \lambda e_i, \quad e_i e_j = e_j e_i$$

if $|i - j| \geq 2$, $(\lambda^{-1} = 4\cos^2 \frac{\pi}{n+3}, \quad n \geq 2; \quad i = 1, 2, ...)$

For fixed n, let $A = Alg\{1, e_1, ..., e_{2n-1}\}$

$A_t = Alg\{1, e_1, ..., e_{n-1}\}, \quad A_s = Alg\{1, e_{n+1}, ..., e_{2n-1}\}$

For $n = 2$: $A = Alg\{1, e_1, e_2, e_3\} \simeq M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$

$A_t = Alg\{1, e_1\} \simeq \mathbb{C} \oplus \mathbb{C}, \quad A_s = Alg\{1, e_3\} \simeq \mathbb{C} \oplus \mathbb{C},$

$$\lambda^{-1} = 4\cos^2 \frac{\pi}{5}$$

Γ-graded Hopf $\star$-algebroid over commutative base :

$\mathbb{G} = (A, B, \Gamma, r, s, \Delta, \varepsilon, S)$, where A and $B = B^{op}$ are unital $\star$-algebras ; there is an action of a group Γ on B, A is $\Gamma \times \Gamma$-graded : $A = \oplus_{\gamma,\gamma' \in \Gamma} A_{\gamma,\gamma'}$; $r \times s : B \otimes B \to A_{e,e}$ is a unital embedding. So ${}_rA_s$ and $A\tilde{\otimes}A := \oplus_{\gamma,\gamma',\gamma''} A_{\gamma,\gamma'} \otimes A_{\gamma',\gamma''} / \{as(b) \otimes a' - a \otimes r(b)a' | a, a' \in A, b \in B\}$ are B-bimodules and unital $\star$-algebras. Coproduct $\Delta : A \to A\tilde{\otimes}A$, counit $\varepsilon : A \to B \rtimes \Gamma$ and antipode $S :_r A_s \to_s A_r$ are B-bimodule and $\star$-algebra maps such that :

$$\Delta(s(b)r(c)) = r(c) \otimes s(b) \text{ for all } b, c \in B,$$

$$(id\tilde{\otimes}\Delta) \circ \Delta = (\Delta\tilde{\otimes}id) \circ \Delta, \quad (id\tilde{\otimes}\varepsilon) \circ \Delta = (\varepsilon\tilde{\otimes}id) \circ \Delta = id,$$

$S(r(b)) = s(b)$, $S(a_{(1)})a_{(2)} = s(\varepsilon(a))$, $a_{(1)}S(a_{(2)}) = r(\varepsilon(a))$ for all $a \in A, b \in B$, and $\Delta \circ S = \Sigma(S\tilde{\otimes}S)\Delta$ (Σ is a "flip").

Integrals and corepresentations

A left integral on $\mathbb{G}$ is a morphism $\phi : (A, r) \to B$ of Γ-graded B-modules s.t. $(id\tilde{\otimes}\phi)\Delta = r \circ \phi$. Similarly a right integral.

$\mathbb{G}$ is called bi-measured if there are a positive map $h : A \to B \otimes B$ which is also a morphism of Γ-graded B-bimodules (a normalized bi-integral) and a positive map $\mu : B \to \mathbb{C}$ such that :

- $\phi := (id \otimes \mu) \circ h$ and $\psi := (\mu \otimes id) \circ h$ are left and right integrals, respectively ;
- $h \circ (r \times s) = id$.
- $\mu(\gamma(bD_\gamma)) = \mu(b), \forall b \in B$, $\gamma \in \Gamma$ for some $D_\gamma \in B$;
- $\nu := (\mu \otimes \mu) \circ h$ is faithful.

A matrix corepresentation of $\mathbb{G}$ is a homogeneous $u \in M_{n_u}(A)$ (i.e., there are $\gamma_1, ..., \gamma_{n_u} \in \Gamma$ such that $u_{i,j} \in A_{\gamma_i,\gamma_j}$ for all i, j) satisfying $(id\tilde{\otimes}\Delta)(u) = u_{12}u_{13}$, $\varepsilon(u_{i,j}) = \delta_{i,j}\gamma_i$, $S(u) = u^{-1}$.

Example : Dynamical $SU_q(2)$ [P.Etingof,A.Varchenko]

Γ-graded Hopf $\star$-algebroid

B is the $\star$-algebra of meromorphic functions on $\mathbb{C}$ with $f^*(\lambda) = \overline{f(\overline{\lambda})}$ and with the action of $\mathbb{Z}$: $k \cdot b(\lambda) := b(\lambda - k)$.

A is the $\mathbb{Z} \times \mathbb{Z}$-graded $\star$-algebra generated by $\alpha \in A_{1,1}, \beta \in A_{1,-1}$, $B \otimes B \subset A_{0,0}$ and relations : $A^*_{k,l} = A_{-k,-l}$,

$\alpha\beta = qF(\mu-1)\beta\alpha,\ \beta\alpha^* = qF(\lambda)\alpha^*\beta,\ \alpha\alpha^*+F(\lambda)\beta^*\beta = 1,\ b(\lambda)\alpha =$

$\alpha b(\lambda+1), b(\lambda)\alpha^* = \alpha^* b(\lambda-1), b(\lambda)\beta = \beta b(\lambda+1), b(\lambda)\beta^* = \beta^* b(\lambda-1),$

where $0 < q < 1$ and $F(\lambda) := \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1}$.

Coproduct : $\Delta(\alpha) = \alpha \otimes \alpha - q^{-1}\beta \otimes \beta^*, \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \alpha^*,$

Antipode : $S(\alpha) = \dfrac{F(\lambda)}{F(\mu)}\alpha^*,\ S(\beta) = -\dfrac{q^{-1}}{F(\mu)}\beta,\ (S \circ \star)^2 = id,$

Counit : $\varepsilon(\alpha) = 1,\ \varepsilon(\beta) = 0.$

Unitary corepresentations [E.Koelink,H.Rosengren]

A B-subbimodule V_n of A generated by $\{(\beta^*)^{n-k}\alpha^k\}_{k=0}^n$ $(n \in \mathbb{N})$

is an A-comodule : $\Delta((\beta^*)^k\alpha^{n-k}) = \Sigma_{j=0}^n t_{kj}^n \tilde{\otimes} (\beta^*)^{n-j}\alpha^j.$

The matrices $t_n = (t_{ij}^n)_{i,j}^n$ define irreducible matrix corepresentations of $SU_q^{dyn}(2)$, $t_{kj}^n \in A$ form a basis in ${}_BA_B$.

We have $t_{kj}^n = P_{n-k}\alpha^{k+j-n}\beta^{k-j}$, where $P_n \in A_{00}$ can be written in terms of Askey-Wilson polynomials.

V_n are unitary : $\Gamma_k(\mu)S(t_{kj}^n)^* = \Gamma_j(\lambda)t_{jk}^n$ for some $\Gamma_k \in B$.

The fusion rule and dimension (same as for $SU_q(2)$ and $SU(2)$) :

$$V_m \otimes_B V_n = \oplus_{s=0}^{min\{m,n\}} V_{m+n-2s}, \quad d(n) := rank_B(V_n) = n + 1.$$

The Haar functional $h : A \to r(B) \otimes s(B)$ sending $f(\lambda)g(\mu)t_{kj}^n$ to $f(\lambda)g(\mu)\delta_{0,n}$ is a normalized bi-integral.

Orthogonality relations : $h(t_{jk}^m(t_{lp}^n)^*) = \delta_{m,n}\delta_{j,l}\delta_{k,p}C(m,j,k,\lambda,\mu,q)$

Unitary representations

- Infinite dimensional [E.Koelink,H.Rosengren] :

$$\pi^\omega(\alpha^*)f(\lambda)e_k = q^k \frac{1-q^{2(\lambda-k+1)}}{1-q^{2(\lambda+1)}} f(\lambda+1)e_k,$$

$\pi^\omega(\beta^*)f(\lambda)e_k = f(\lambda-1)e_{k+1},\ \pi^\omega(r(g))f(\lambda)e_k = g(\lambda-\omega-2k)f(\lambda)e_k,$

$\pi^\omega(s(g))f(\lambda)e_k = g(\mu)f(\lambda)e_k,\ \pi^\omega(a^*) = \pi^\omega(a)^*$, for all $a \in A$

on $V = \oplus_{k\in\mathbb{N}} Be_k$ with scalar product $< fe_k, ge_l >=$

$$= \delta_{k,l} \int_{\mathbb{R}} f(\lambda)\overline{g(\lambda)} \frac{(q^2, q^{2\omega}; q^2)_k}{(q^{2(\lambda-k+1)}, q^{2(\omega-\lambda+k-1)}; q^2)_k} d\lambda, \quad \text{where}$$

$\omega \in \mathbb{R}$, $(a,b;q^2)_k := (a;q^2)_k(b;q^2)_k$, $(a;q^2)_k := \Pi_{j=0}^{k-1}(1-aq^{2j})$.

- "1-dimensional "$\star$-homomorphisms $A \to B \rtimes \mathbb{Z}$:

$$\pi_k(\alpha) = (exp(2\pi ki\lambda), 1),\ \pi_k(\delta) = (exp(-2\pi ki\lambda), -1),$$

$$\pi_k(\beta) = \pi_k(\gamma) = 0,\ \pi_k(b\otimes b') = bb',\ \text{for all } b, b' \in B, k \in \mathbb{Z}.$$

Towards C^*-algebraic $SU_q^{dyn}(2)$ (variation on a theme by T.Timmermann)

1. Replace B by $\tilde{B} = M(B_0)$, where $B_0 := \{f \in C_0(\mathbb{R}) | f|_{\mathbb{Z}} = 0\}$. $F^{\pm 1}(\lambda - k)$ $(k \in \mathbb{Z},\ \lambda \in Q = \mathbb{R} \backslash \mathbb{Z})$ can be viewed as elements affiliated with the C^*-algebra B_0.

2. Put $\nu := (\mu \otimes \mu) \circ h$, where μ is a probability measure with $supp(\mu) = \mathbb{R}$, and $D_k(\lambda) := \frac{d\mu \circ T_k}{d\mu} \in \tilde{B}$ $(T_k : b(\lambda) = b(\lambda - k))$.

Define $\Delta(b(\lambda)c(\mu)) := b(\lambda) \otimes_{\tilde{B}} c(\mu),\ \forall b, c \in \tilde{B}$.

3. Define the fundamental unitary $V : H \otimes_{\tilde{B}} H \to H \otimes_{\tilde{B}} H$, where $H := L^2(A, \nu)$, by

$$V(x \otimes_{\tilde{B}} y) := S^{-1}(r(D_{-k}^{-1/2})y_{(1)})x \otimes_{\tilde{B}} y_{(2)}, \text{ if } y \in A_{k,l}.$$

Using V, one shows that $\pi_\nu : A \to L(H)$ such that $\pi_\nu(a)x := ax$ is a $\star$-representation of A. Define $A_{red} := \overline{\pi_\nu(A)}$.

Remark If $a \in A$, let $\hat{a}$ be a linear form on A given by $\hat{a}(x) := \nu(S(a)x)$, and define right convolution $x \star \hat{a} := x_{(2)} r(h(S(a)x_{(1)})$, where $x \in A$. Then $\hat{A}$ is a unital $\star$-algebra with $\hat{x} \star \hat{y} := \widehat{x \star \hat{y}}$, $(\hat{x})^* := \widehat{S(x)^*}$. Moreover, $\rho_\nu : \hat{A} \to L(H)$ such that $\rho_\nu(a)x := := x \star \hat{a}$ is a $\star$-representation. Let us denote $\hat{A}_{red} := \overline{\rho_\nu(\hat{A})}$.

3. The Pentagonal relation

$$V_{23} V_{12} = V_{12} V_{13} V_{23}$$

allows to equip A_{red} and $\hat{A}_{red}$ with coproducts :

$$\Delta(\pi_\nu(a)) = V^*(id \otimes_{\tilde{B}} \pi_\nu(a))V,$$

$$\widehat{\Delta}(\rho_\nu(\hat{a})) = \Sigma V(\rho_\nu(\hat{a}) \otimes_{\tilde{B}} id)V^*\Sigma,$$

they become Hopf C^*-bimodules over $\tilde{B}$.

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