Linear isometries of Hilbert C^*-modules

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Complex Hilbert C^*-module

A : **complex** C^*-algebra.

Definition

V : **complex** Hilbert A-module if V is a **(right)** A-module,
$\exists \langle \cdot, \cdot \rangle : V \times V \to \mathbf{A}$ such that

1. $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle, \ \forall \ x, y, z \in V, \lambda \in \mathbb{C}$;
2. $\langle x, ya \rangle = \langle x, y \rangle a, \ \forall \ x, y \in V, \ a \in A$;
3. $\langle x, y \rangle^* = \langle y, x \rangle, \ \forall \ x, y \in V$;
4. $\langle x, x \rangle \geq 0, \ \forall \ x \in V$; $\langle x, x \rangle = 0$ iff $x = 0$;
5. V is **complete** with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

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- $\overline{H} = \{\overline{h} : h \in H\}$: conjugate linear isomorphic to H.
  Then **$\overline{H}$ is a Hilbert $\mathbb{C}$-module** with $\overline{h}\cdot\lambda = \overline{\overline{\lambda}h}$ and $\langle \overline{h}, \overline{k}\rangle = (h, k)$.

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- $\overline{H} = \{\overline{h} : h \in H\}$: conjugate linear isomorphic to H.
  Then $\overline{H}$ **is a Hilbert $\mathbb{C}$-module** with $\overline{h}\cdot\lambda = \overline{\overline{\lambda}h}$ and $\langle \overline{h}, \overline{k}\rangle = (h, k)$.
- $K(H)$: C^*-algebra of compact operators on H.
  Then $\overline{H}$ **is a Hilbert $K(H)$-module**, denoted by $\overline{H}_K$, with

$$\overline{h}\cdot T = \overline{T^*(h)} \text{ and } \langle \overline{h}, \overline{k}\rangle = h \otimes k.$$

Here $h\otimes k$ is the rank-one operator defined by $h\otimes k(x) = (x, k)h$.

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- H, K : complex Hilbert spaces.
  every surjective $\mathbb{C}$-linear isometry $T : H \to K$ is unitary, i.e.,
  $$\langle Th, Tk \rangle = \langle h, k \rangle.$$

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$T : V \to W$ is a surjective $\mathbb{C}$-linear isometry. Then

T **is A-linear**, $T(xa) = (Tx)a$, $\Rightarrow$ T **is unitary**, $\langle Tx, Ty\rangle = \langle x, y\rangle$.

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$T : V \to W$ is a surjective $\mathbb{C}$-linear isometry.

$\alpha : A \to B$: $*$-isomorphism. Then

$$T \text{ is a module map}, \; T(xa) = (Tx)\alpha(a),$$

if and only if

$$T \text{ is unitary}, \; \langle Tx, Ty \rangle = \alpha(\langle x, y \rangle).$$

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Lemma (Banach-Stone Theorem)

$T : C_0(X, H) \to C_0(Y, K)$ a surjective linear isometry.
Then $\exists\ \varphi : Y \to X$ a homeo., $h_y : H \to K$: unitary, $\forall\ y \in Y$ such that

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equivalently, T is unitary $\langle Tf, Tg\rangle = \alpha(\langle f, g\rangle)$.

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  Is every surjective linear isometry $T : V \to W$ a unitary, equivalently, module map?
- Yes, if A and B are commutative.
- No, if one of them is noncommutative.

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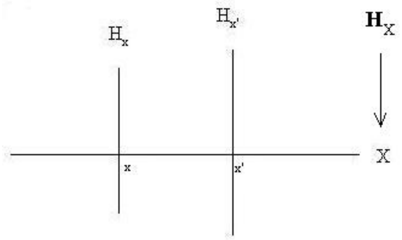
The identity map $I : \overline{H}_K \to \overline{H}$ is a surjective linear isometry.
However, $\nexists$ $*$-isomorphism between $K(H)$ and H if $\dim H > 1$.



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  $\mathbb{H}_X$: topological space.
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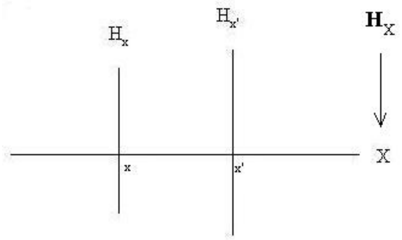

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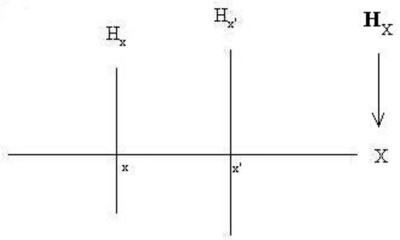

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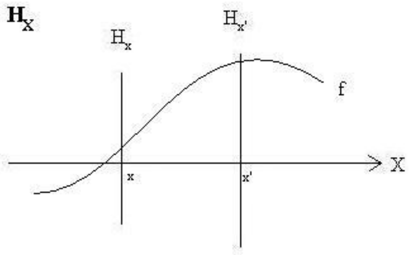
  (1) $\cdot, +, \|\cdot\|$ on $\mathbb{H}_X$ are continuous wherever they are defined.
  (2) If $x \in X$ and $\{h_i\}$ is a net in $\mathbb{H}_X$ such that $\|h_i\| \to 0$ and $\pi(h_i) \to x$ in X, then $h_i \to 0_x$ (the zero element of H_x) in $\mathbb{H}_X$.



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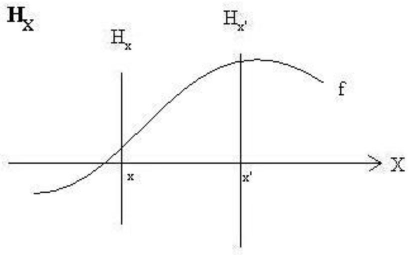

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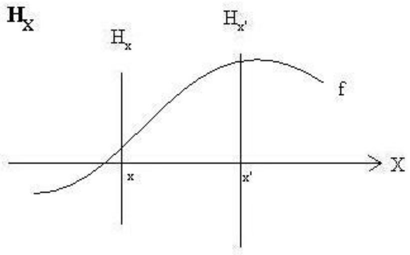

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- A C_0-section $f : X \to \mathbb{H}_X$ is a conti. section vanishing at infinity.
- $C_0(X, \mathbb{H}_X)$: Banach space of C_0-sections.

Theorem

$\langle \mathbb{H}_X, \pi_X \rangle \cong \langle \mathbb{H}_Y, \pi_Y \rangle$ *if and only if* $C_0(X, \mathbb{H}_X) \cong C_0(Y, \mathbb{H}_Y)$.

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$T : C_0(X, \mathbb{H}_X) \to C_0(Y, \mathbb{H}_Y)$ *: surjective linear isometry.*
Then $\exists\ \varphi : Y \to X$ *: homeomorphism,* $h_y : H_{\varphi(y)} \to H_y$ *: unitary, such that*

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The bundle isomorphism is defined by

$$\Phi = (h_y)_{y \in Y},\ \text{i.e., } \Phi|_{H_{\varphi(y)}} = h_y.$$

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Lemma

V : Hilbert $C_0(X)$-module.
Then $V \cong C_0(X, \mathbb{H}_X)$, for some Hilbert bundle $\langle \mathbb{H}_X, \pi_X \rangle$ over X, i.e., $\exists$ a unitary map

$$\widehat{\ } : V \to C_0(X, \mathbb{H}_X)$$

$$\langle \widehat{u}, \widehat{v} \rangle = \langle u, v \rangle \quad \text{and} \quad \widehat{v\phi} = \widehat{v}\phi.$$

[0] M. J. Dupré and R. M. Gillette, *Banach bundles, Banach modules and automorphisms of C^*-algebras*, Research Notes in Mathematics 92, Pitman, 1983.

Theorem

V : Hilbert $C_0(X)$-module. W : Hilbert $C_0(Y)$-module.
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Noncommutative cases

- V : Hilbert A-module. V is *full* if

$$\langle V, V\rangle = \mathrm{span}\{\langle u, v\rangle : u, v \in V\} \text{ is dense in } A.$$

Lemma

*V, W : complex **full** Hilbert A, B-modules, respectively.*
*$T : V \to W$: surjective linear **2-isometry**.*
Then $\exists$ a $*$-isomorphism $\alpha : A \to B$ such that

$$T \text{ is unitary and a module map.}$$

[0]B. Solel, Isometries of Hilbert C^*-modules, *Trans. Amer. Math. Soc.* **553** (2001), 4637-4660.

V : Hilbert A-module.
Then $M_n(V)$: Hilbert $M_n(A)$-module with the following module action and inner product.

$$[x_{ij}][a_{ij}] = [z_{ij}], \quad z_{ij} = \sum_{k=1}^{n} x_{ik}a_{kj}$$

$$\langle [x_{ij}], [y_{ij}] \rangle = [b_{ij}], \quad b_{ij} = \sum_{k=1}^{n} \langle x_{ki}, y_{kj} \rangle,$$

for all $[x_{ij}], [y_{ij}]$ in $M_n(V)$, $[a_{ij}]$ in $M_n(A)$.

- $T : V \to W$: linear map.
  Define $T_n : M_n(V) \to M_n(W)$ by
  $$T_n((x_{ij})_{ij}) = (T(x_{ij}))_{ij}.$$
- T : *n-isometry* if T_n is a isometry.
- T : *complete isometry* if all T_n are isometries.

JB*-triples

- V : complex vector space.
  If $\exists \{x, y, z\} : V^3 \to V$: linear in x and z, conjugate linear in y, and satisfies the following identities:
  (1) $\{x, y, z\} = \{z, y, x\}$;
  (2) $\{x, y, \{z, u, v\}\} =$
  $\{\{x, y, z\}, u, v\} - \{z, \{y, x, u\}, v\} + \{z, u, \{x, y, v\}\}$.
  Then V is called *complex Jordan triple*,
  $\{x, y, z\}$ is called **Jordan triple product**.

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  Then V is called *complex Jordan triple*,
  $\{x, y, z\}$ is called **Jordan triple product**.
- A complex Banach space $(V, \|\cdot\|)$ **: JB*-triple** if it is a complex Jordan triple with a continuous triple product and $a\Box a$, defined by $a\Box a : V \to V, \quad b \mapsto \{a, a, b\}$, satisfies the following conditions:
  (a) $a\Box a$ is a hermitian operator on V;
  (b) $a\Box a$ has nonnegative spectrum;
  (c) $\|a\Box a\| = \|a\|^2$.

Lemma

Let T be a linear bijective map between JB*-triples. Then T is a isometry if and only if it preserves Jordan triple products,

$$T\{x, y, z\} = \{Tx, Ty, Tz\}.$$

[0] C.-H. Chu, *Jordan Structures in Geometry and Analysis*, Cambridge University Press, 2012.

[0] J. M. Isidro, Holomorphic automorphisms of the unit balls of Hilbert C^*-modules. *Glasg. Math. J.* **45** (2003), no. 2, 249-262.

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Every complex Hilbert C^*-module is a JB*-triple with Jordan triple product $\{x, y, z\} = \frac{1}{2}(x\langle y, z\rangle + z\langle y, x\rangle)$.

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V, W : complex Hilbert C^*-modules.

$T : V \to W$: surjective linear isometry. Then

$$T(x\langle x,x\rangle) = Tx\langle Tx,Tx\rangle.$$

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- If T is a 2-isometry, then $T_2 : M_2(V) \to M_2(V)$: isometry. T_2 preserves Jordan triple products

$$T_2(u\langle u,u\rangle) = T_2u\langle T_2u, T_2u\rangle, \quad \forall\ u \in M_2(V). \tag{1}$$

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- Let $u = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$ in $M_2(V)$. Then

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  $\Rightarrow T$ preserves **ternary (TRO) products** $T(x\langle y,z\rangle) = Tx\langle Ty, Tz\rangle$.

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  V and W are full,

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Summary

Theorem

A, B : complex C^*-algebras.

V, W : complex full Hilbert A, B-modules, respectively.

$T : V \to W$: surjective linear isometry. Then TFAE.

1. *T : 2-isometry.*
2. *T : complete isometry.*
3. *$\langle Tx, Ty \rangle = \alpha(\langle x, y \rangle)$, for some $*$-isomorphism $\alpha : A \to B$.*
4. *$T(xa) = (Tx)\alpha(a)$, for some $*$-isomorphism $\alpha : A \to B$.*
5. *$T(x\langle y, z \rangle) = Tx\langle Ty, Tz \rangle$.*

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If A and B are **commutative***, the five statements hold automatically.*

Recall

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- T : isometry $\Rightarrow T$: unitary if A and B are commutative.

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Can we drop the linearity of T?

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A, B : complex C^*-algebras.

V, W : full Hilbert A, B-modules.

$T : V \to W$: surjective linear isometry.

- T : module map, $\Rightarrow T$: unitary.
- T : 2-isometry $\Rightarrow T$: unitary.
- T : isometry $\Rightarrow T$: unitary if A and B are commutative.

Can we drop the linearity of T?

Lemma (Mazur-Ulam Theorem)

An surjective isometry $T : V \to W$ of a normed linear space V onto another normed linear space W with $T(0) = 0$ is **real linear**.

Real C^*-algebra

- Real algebra A : $a + b, ab, \lambda a \in A$, $\forall a, b \in A$, $\lambda \in \mathbb{R}$.

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  $A_c = A + iA = \{a + ib : a, b \in A\}$.
  Is there a norm $\|\cdot\|_c$ on A_c such that
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  (2) $\|a+ib\|_c = \|a-ib\|_c$?

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  Define $\overline{a+ib} = a - ib$. Then $A = \{a_c \in A_c : \overline{a_c} = a_c\}$.

Lemma

Every real Banach algebra has a unique (up to equivalence) complexification.

[0]B. Li, *Real operator algebras*, World Scientific Publishing Co., Inc., River Edge, N. J., 2003.

- A real Banach $*$-algebra A is a real Banach algebra with a (real) linear operator $* : A \to A$ such that $(ab)^* = b^* a^*$ and $a^{**} = a$.

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  Define $(a + ib)^* = a^* - ib^*$.
  Then A_c is a complex Banach $*$-algebra.

Definition

A real Banach $*$-algebra A is called a *real C^*-algebra* if we can extend the norm of A to $A_c = A + iA$ such that A_c is a complex C^*-algebra.

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Lemma

Let A be a real Banach $*$-algebra. Then TFAE.

1. *A is a real C^*-algebra;*
2. *A can be isometrically $*$-isomorphic to a norm closed $*$-subalgebra of $B(H)$ on a real Hilbert space H;*
3. *$1 + a^*a$ is invertible $\widetilde{A}$ and $\|a^*a\| = \|a\|^2$, for all a in A.*

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- $\mathbb{C}$ with $z^* = z$ is a real Banach $*$-algebra such that $|z^*z| = |z|^2$. However, $1 + i^*i = 0$ is not invertible.

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Example

- H : real Hilbert space, $(h, k) \in \mathbb{R}$.
  $H_c = H + iH$: complex Hilbert space with inner product
  $(h + ik, x + iy) = (h, x) + (k, y) + i(k, x) - i(h, y)$.
  $\Rightarrow \|h + ik\|^2 = \|h - ik\|^2 = \|h\|^2 + \|k\|^2$.
- For T in $B(H)$, define $T_c \in B(H_c)$ by $T_c(h + ik) = T(h) + iT(k)$.
  Then

$$\begin{aligned}\|T_c(h+ik)\|^2 &= \|T(h) + iT(k)\|^2 = \|T(h)\|^2 + \|T(k)\|^2 \\ &\leq \|T\|^2(\|h\|^2 + \|k\|^2) = \|T\|^2\|h + ik\|^2.\end{aligned}$$

  $\Rightarrow \|T_c\| = \|T\|$, $\|T + iS\| = \|T - iS\|$.
  $\Rightarrow B(H_c) \cong B(H) + iB(H)$.

Example

- X : locally compact Hausdorff space.
  $\sigma : X \to X$: a homeomorphism, $\sigma^2(x) = x$, $\forall\, x \in X$.
  $C_0(X, \sigma) = \{f \in C_0(X) : f(\sigma(x)) = \overline{f(x)}\}$.

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- For f in $C_0(X)$, define

$$g = \frac{1}{2}(f + \overline{f \circ \sigma}) \quad \text{and} \quad h = \frac{1}{2i}(f - \overline{f \circ \sigma}).$$

  Then $g, h \in C_0(X, \sigma)$ and $f = g + ih$.

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- $C_0(X) = C_0(X, \sigma) + iC_0(X, \sigma)$.
- Every commutative real C^*-algebra is of the form $C_0(X, \sigma)$ up to a $*$-isomorphism.

Real Hilbert C^*-modules

- A : **real** C^*-algebra.

Definition

V : **real** Hilbert A-module if V is a A-module,
$\exists\ \langle\cdot,\cdot\rangle : V \times V \to A$ such that

1. $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle,\ \forall\ x, y, z \in V, \lambda \in \mathbb{R}$;
2. $\langle x, ya\rangle = \langle x, y\rangle a,\ \forall\ x, y \in V,\ a \in A$;
3. $\langle x, y\rangle^* = \langle y, x\rangle,\ \forall\ x, y \in V$;
4. $\langle x, x\rangle \geq 0,\ \forall\ x \in V$; $\langle x, x\rangle = 0$ iff $x = 0$;
5. V is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

Banach-Stone Theorem for real C^*-algebras

Lemma

$T : C_0(X, \sigma) \to C_0(Y, \tau)$: *surjective linear isometry.*
Then $\exists\ \varphi : Y \to X$: *homeomorphism,*
$h \in C(Y, \tau)$ *with* $|h(y)| = 1$, *such that*

$$\sigma \circ \varphi = \varphi \circ \tau \quad \text{and} \quad Tf(y) = h(y) f(\varphi(y)).$$

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \tau \downarrow & & \downarrow \sigma \\ Y & \xrightarrow{\varphi} & X \end{array}$$

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$$\Rightarrow \ \langle Tf, Tg\rangle = \overline{(Tf)}(Tg) = \overline{(f\circ\varphi)}(g\circ\varphi) = \langle f, g\rangle \circ \varphi = \alpha(\langle f, g\rangle).$$

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  is a real Hilbert $C_0(X, \sigma)$-module with

$$(f\psi)(x) = f(x)\psi(x), \quad f \in C_0(X, \mathbb{H}_X, \sigma, -), \psi \in C_0(X, \sigma)$$

and

$$\langle f, g \rangle(x) = (f(x), g(x)), \quad f, g \in C_0(X, \mathbb{H}_X, \sigma, -).$$

Theorem

V : real Hilbert A-module.

Then $V_c = V + iV$: complex Hilbert $A_c = (A + iA)$-module.

Sketch of proof:

- $(x + iy)(a + ib) := (xa - yb) + i(xb + ya)$.
  $\langle u + iv, x + iy \rangle := (\langle u, x \rangle + \langle v, y \rangle) + i(\langle u, y \rangle - \langle v, x \rangle)$.

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- To see $\langle x + iy, x + iy\rangle \geq 0$.
  Note $\langle x + iy, x + iy\rangle = \langle x + iy, x + iy\rangle^*$.
  Check $f(\langle u + iv, x + iy\rangle) \geq 0$, $\forall$ positive linear functional f on A_c.

- $f(\langle x, y\rangle) = f(\langle y, x\rangle^*) = \overline{f(\langle y, x\rangle)}$.
- $|f(\langle x, y\rangle)|^2 \le f(\langle x, x\rangle) f(\langle y, y\rangle)$.

$$
\begin{aligned}
& f(\langle x+iy, x+iy\rangle) \\
&= f(\langle x, x\rangle) + f(\langle y, y\rangle) + if(\langle x, y\rangle) - if(\langle y, x\rangle) \\
&= f(\langle x, x\rangle) + f(\langle y, y\rangle) + 2\,\mathrm{Re}\, if(\langle x, y\rangle) \\
&\ge f(\langle x, x\rangle) + f(\langle y, y\rangle) - 2|f(\langle x, y\rangle)| \\
&\ge f(\langle x, x\rangle) + f(\langle y, y\rangle) - 2f(\langle x, x\rangle)^{1/2} f(\langle y, y\rangle)^{1/2} \\
&= (f(\langle x, x\rangle)^{1/2} - f(\langle y, y\rangle)^{1/2})^2 \ge 0.
\end{aligned}
$$

Theorem

V : real Hilbert $C_0(X,\sigma)$-module.
$\exists$ conjugate linear isometric isomorphisms $- : H_x \to H_{\sigma(x)}$ such that $V \cong C_0(X, \mathbb{H}_X, \sigma, -)$.

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Let $I_x = \{f \in C_0(X) : f(x) = 0\}$,

$H_x := V_c/V_cI_x$ with $(u_c + V_cI_x, v_c + V_cI_x) = \langle u_c, v_c\rangle(x)$.

$V_c \cong C_0(X, \mathbb{H}_X), \; v_c(x) = v_c + VI_x$.

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Then $\exists\ \varphi : Y \to X$: homeomorphism, $h_y : H_{\varphi(y)} \to H_y$: unitary, s.t.
$\sigma \circ \varphi = \varphi \circ \tau$ and $Tf(y) = h_y(f(\varphi(y)))$.

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$$\begin{aligned} \Rightarrow\ \langle Tf,Tg\rangle &= \langle h_y(f(\varphi(y))),h_y(g(\varphi(y)))\rangle = \langle f(\varphi(y)),g(\varphi(y))\rangle \\ &= \langle f,g\rangle(\varphi(y)) = \alpha(\langle f,g\rangle)(y). \end{aligned}$$

General case

- V : real vector space.
  $\{x, y, z\} : V^3 \to V$: trilinear and satisfies the following identities:
  - $\{x, y, z\} = \{z, y, x\}$;
  - $\{x, y, \{z, u, v\}\} =$
    $\{\{x, y, z\}, u, v\} - \{z, \{y, x, u\}, v\} + \{z, u, \{x, y, v\}\}$.

  Then V is called *real Jordan triple*.
- If $V_c = V + iV$ is furnished with the triple product
  $\{x + iu, y + iv, x + iu\}_c = (\{x, y, x\} - \{u, y, u\} + 2\{x, v, u\}) +$
  $i(-\{x, v, x\} + \{u, v, u\} + 2\{x, y, u\})$. Then $(V_c, \{\cdot, \cdot, \cdot\}_c)$ is a complex Jordan triple, called the *complexification* of $(V, \{\cdot, \cdot, \cdot\})$.

Definition

A real Banach space V is called a *real JB*-triple* if it is a real Jordan triple such that its complexification $(V_c, \{\cdot,\cdot,\cdot\}_h)$ can be normed to become a JB*-triple.

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Theorem

Every real Hilbert C^*-module is a real JB*-triple with Jordan triple product $\{x, y, z\} = \frac{1}{2}(x\langle y, z\rangle + z\langle y, z\rangle)$.

V : Hilbert A-module. $\Rightarrow V_c$: Hilbert A_c-module which is a JB*-triple.

Lemma

A, B : real C^*-algebras.

$T : V \to W$: a bounded linear bijective map.

Then T is a isometry if and only if it preserves Jordan triple products.

Jordan triple product of a C^*-algebra : $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$.

[0] C.-H. Chu, *Jordan Structures in Geometry and Analysis*, Cambridge University Press, 2012.

Example

- $M_{1,2}(\mathbb{C})$: real JB*-triple with triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$.
  $T : M_{1,2}(\mathbb{C}) \to M_{1,2}(\mathbb{C})$, $T(\alpha + i\beta, \gamma + i\delta) = (\alpha + i\gamma, \beta + i\delta)$.
- T is a surjective real linear isometry (it is not complex linear).
  But T does not preserve Jordan triple products.
  For example, let $x = (1 + i, 0), y = (0, 1)$. Then

$$(0, 0) = T\{x, y, x\} \neq \{Tx, Ty, Tx\} = -(i, i).$$

Lemma

V, W : real JB-triples.*

$T : V \to W$: a bounded linear bijective map.

Then

(1) *T is a isometry if it preserves Jordan triple products.*

(2) *If T is a isometry then*

$$T(\{x, x, x\}) = \{Tx, Tx, Tx\},$$

for all x, y, z in V.

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- If T is a 2-isometry, then $T_2 : M_2(V) \to M_2(V)$: isometry.
  T_2 preserves Jordan triple products

$$T_2(u\langle u,u\rangle) = T_2u\langle T_2u, T_2u\rangle, \quad \forall\, u \in M_2(V). \tag{2}$$

- Let $u = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$ in $M_2(V)$.
  Then

$$u\langle u,u\rangle = \begin{pmatrix} * & x\langle y,z\rangle \\ * & * \end{pmatrix}.$$

  The equation (2) becomes

$$\begin{pmatrix} * & T(x\langle y,z\rangle) \\ * & * \end{pmatrix} = \begin{pmatrix} * & Tx\langle Ty,Tz\rangle \\ * & * \end{pmatrix}.$$

  $\Rightarrow T$ preserves ternary (TRO) products $T(x\langle y,z\rangle) = Tx\langle Ty,Tz\rangle$.

- T : 2-isometry $\Rightarrow T(x\langle y,z\rangle) = Tx\langle Ty, Tz\rangle \Rightarrow T$: unitary

  Define $\alpha : \langle V, V\rangle \to \langle W, W\rangle$ by

$$\alpha(\sum_{i=i}^{n} c_i\langle x_i, y_i\rangle) := \sum_{i=i}^{n} c_i\langle Tx_i, Ty_i\rangle.$$

  V and W are full,

  $\alpha : A \to B$ is a $*$-isomorphism such that

$$\langle Tx, Ty\rangle = \alpha(\langle x, y\rangle).$$

  Conversely, suppose T is unitary.

$$\begin{aligned}\langle Tw, T(x\langle y,z\rangle)\rangle &= \alpha(\langle w, x\langle y,z\rangle\rangle) = \alpha(\langle w,x\rangle\langle y,z\rangle) \\ &= \alpha(\langle w,x\rangle)\alpha(\langle y,z\rangle) = \langle Tw, Tx\rangle\langle Ty, Tz\rangle \\ &= \langle Tw, Tx\langle Ty, Tz\rangle\rangle.\end{aligned}$$

  $\Rightarrow T(x\langle y,z\rangle) = Tx\langle Ty, Tz\rangle$.

  $\Rightarrow$ Each $T_n : M_n(V) \to M_n(W)$ preserves Jordan triple products.

  $\Rightarrow T_n$ is a isometry, $\forall n. \Rightarrow T$ is a complete isometry.

Summary

V, W : real Hilbert A, B-modules, respectively.
$T : V \to W$: surjective linear isometry. Then TFAE.

(a) T : 2-isometry.

(b) T : complete isometry.

(c) $\langle Tx, Ty\rangle = \alpha(\langle x, y\rangle)$, for some $*$-isomorphism $\alpha : A \to B$.

(d) $T(xa) = (Tx)\alpha(a)$, for some $*$-isomorphism $\alpha : A \to B$.

(e) $T(x\langle y, z\rangle) = Tx\langle Ty, Tz\rangle$.

Summary

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If A and B are commutative, these four statements hold automatically.

Thank you for your attention