Big Data Big Bias Small Surprise

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Outline of Presentation

Proposed Estimation Strategies

Asymptotic and Simulation Study

Applications

Envoi

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Outline of Presentation

Classical Linear Model

Consider a classical linear model with observed response variable y_i and covariates $\mathbf{x}_i = (x_{i1}, \cdots, x_{ip_n})'$ as follows,

$$y_i = \mathbf{x}_i' \boldsymbol{\beta}_n + \epsilon_i, \quad 1 \leq i \leq n,$$

where $\boldsymbol{\beta}_n = (\beta_1, \cdots, \beta_{p_n})'$ is a p_n-dimensional vector of the unknown parameters, and ϵ_i's are independent and identically distributed with center 0 and variance σ^2.

Subscript n in p_n indicates that the number of coefficients may increase with the sample size n.

Model Selection & Estimation Problem

Candidate Full Model Estimation

A Great Deal of Redundancy in the Candidate Full Model

Too Many Nuisance Regression Parameters

Candidate Full Model is Sparse

Candidate Subspace

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We want to estimate β when it is plausible that β lie in the subspace

$$\mathbf{H}\beta = \mathbf{h}$$

- Human Eye: Uncertain Prior Information (UPI)
- Machine Eye: Auxiliary Information (AE)

$$UPI \text{ or } AI: \ \mathbf{H}\beta = \mathbf{h}$$

In many applications it is assumed that model is sparse, i.e. $\beta = (\beta_1', \beta_2')', \quad \beta_2 = \mathbf{0}.$

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Candidate Full Model Estimation

- Maximum Likelihood
- Least Square
- Ridge regression Or any other

Candidate Submodel Estimation

$$\hat{\beta}^{SM} = \hat{\beta}^{FM} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'(\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\beta}^{FM} - \mathbf{h}).$$

- An interesting application of the restriction is that β can be partitioned as $\beta = (\beta_1', \beta_2')'$, if model is sparse, then $\beta_2 = \mathbf{0}$

Sparsity is the Name of the Game?

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Classical Model Selection

Preliminary Testing

$$H_0 : \mathbf{H}\beta = \mathbf{h} \qquad H_a : \mathbf{H}\beta \neq \mathbf{h}$$

Test Statistics

$$T_n = \frac{(\mathbf{H}\hat{\beta}^{FM} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\beta}^{FM} - \mathbf{h})}{s_e^2}, \qquad (1)$$

where

$$s_e^2 = \frac{(\mathbf{Y} - \mathbf{X}\hat{\beta}^{FM})'(\mathbf{Y} - \mathbf{X}\hat{\beta}^{FM})}{n - p}$$

Pretest Estimation Strategy

The pretest estimator (PTE) of β based on $\hat{\beta}^{FM}$ and $\hat{\beta}^{SM}$ is defined as

$$\hat{\beta}^{PT} = \hat{\beta}^{FM} - (\hat{\beta}^{FM} - \hat{\beta}^{SM}) I(T_n \leq \chi^2_{p_2,\alpha}), \quad p_2 \geq 1,$$

$I(A)$ is an indicator function of a set A and $\chi^2_{p_2,\alpha}$ is the α-level

critical value of the distribution of T_n under H_0.

Shrinkage Estimation Strategy

$$\hat{\beta}^{S} = \hat{\beta}^{SM} + \left(1 - (p_2 - 2)\, T_n^{-1}\right) (\hat{\beta}^{FM} - \hat{\beta}^{SM}), \quad p_2 \geq 3,$$

Possible over-shrinking problem is defined as

$$\hat{\beta}^{S+} = \hat{\beta}^{SM} + \left(1 - (p_2 - 2)\, T_n^{-1}\right)^{+} (\hat{\beta}^{FM} - \hat{\beta}^{SM}),$$

where $z^{+} = max(0, z)$.

Estimation Strategies

Shrinkage Estimation Strategy

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Executive Summary

- Bancroft (1944) suggested two problems on preliminary test strategy.
  - Data pooling problem based on a preliminary test. This stream followed by a host of researchers.
  - Model selection problem in linear regression model based on a preliminary test.
- Stein (1956, 1961) developed highly efficient shrinkage estimators in balanced designs. Most statisticians have ignored these (perhaps due to lack of understanding)
- Modern regularization estimation strategies based on penalized least squares with penalties extend Stein's procedures powerfully.

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Penalty Estimation Strategy

- The penalty estimators are members of the penalized least squares (PLS) family and they are obtained by optimizing a quadratic function subject to a penalty.
- PLS estimation provides a generalization of both nonparametric least squares and weighted projection estimators.
- A popular version of the PLS is given by Tikhonov (1963) regularization.
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- For a given penalty function $\pi(\cdot)$ and regularization parameter λ, the general form of the objective function can be written as

$$\phi(\beta) = (\boldsymbol{y} - \boldsymbol{X}\beta)^T(\boldsymbol{y} - \boldsymbol{X}\beta) + \lambda\pi(\beta),$$

- Penalty function is of the form

$$\pi(\beta) = \sum_{j=1}^{p} |\beta_j|^\gamma, \ \gamma > 0. \tag{2}$$

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For $\gamma = 2$, we have ridge estimates which are obtained by minimizing the penalized residual sum of squares

$$\hat{\beta}^{\text{ridge}} = \arg\min_{\beta} \left\| \boldsymbol{y} - \sum_{j=1}^{p} \boldsymbol{X}_j \beta_j \right\|^2 + \lambda \sum_{j=1}^{p} ||\beta_j||^2, \qquad (3)$$

λ is the tuning parameter which controls the amount of shrinkage and $|| \cdot || = || \cdot ||_2$ is the L_2 norm.

Penalty Estimation Strategy

- For $\gamma < 2$, it shrinks the coefficient towards zero, and depending on the value of λ, it sets some of the coefficients to exactly zero.
- The procedure combines variable selection and shrinking of the coefficients of a penalized regression.
- An important member of the penalized least squares family is the L_1 penalized least squares estimator, which is obtained when $\gamma = 1$.
- This is known as the Least Absolute Shrinkage and Selection Operator (LASSO): Tibshirani(1996)

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- LASSO is closely related to the ridge regression and its solutions are similarly obtained by replacing the squared penalty $||\beta_j||^2$ in the ridge solution (**??**) with the absolute penalty $||\beta_j||_1$ in the LASSO–

$$\hat{\beta}^{\text{LASSO}} = \arg\min_{\beta} \left\| \boldsymbol{y} - \sum_{j=1}^{p} \boldsymbol{X}_j \beta_j \right\|^2 + \lambda \sum_{j=1}^{p} ||\beta_j||_1. \quad (4)$$

Good Strategy if Model is Sparse

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Algorithm, Algorithm, Algorithm

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- LARS, least angle regression provides a clever and very efficient algorithm of computing the complete LASSO sequence of solutions as s is varied from 0 to ∞
- Friedman, et al. (2007, 2008) and Wu and Lange developed the coordinate descent (CD) algorithm for penalized linear regression and penalized logistic regression and was shown to gain computational superiority.
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Penalty Estimation

Family Ever Growing!!

Adaptive LASSO

Elastic Net Penalty

Minimax Concave Penalty

SCAD

Extension and Comparison with non-penalty Estimators

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- Fallahpour, Ahmed and Doksum (2010) partially linear models with Random Coefficient autoregressive Errors.
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- Adaptive LASSO, SCAD, and MCP are Oracle (asymptoticaly).
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- Nonzero coefficients are big enough to to be separated from zero ones.

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Innate Difficulties: Can Signals be Septated from Noise?

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Shrinkage Estimation for Big Data

- The classical shrinkage estimation methods are limited to fixed p.
- The asymptotic results depend heavily on a maximum likelihood full estimation with component-wise consistency at rate of $\sqrt{n}$.
- When $p_n > n$, a component-wise consistent estimator of β_n is not available since β_n is not identifiable.
- Here β_n is not identifiable in the sense that there always exist two different estimations of β_n, $\beta_n^{(1)}$ and $\beta_n^{(2)}$, such that $\mathbf{x}_i'\beta_n^{(1)} = \mathbf{x}_i'\beta_n^{(2)}$ for $1 \leq i \leq n$.

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- we write the p_n-dimensional coefficients vector $\boldsymbol{\beta}_n = (\boldsymbol{\beta}'_{1n}, \boldsymbol{\beta}'_{2n})'$,, where $\boldsymbol{\beta}_{1n}$ is the coefficient vector for main covariates, $\boldsymbol{\beta}_{2n}$ include all nuisance parameters.
- Sub-vectors $\boldsymbol{\beta}_{1n}$, $\boldsymbol{\beta}_{2n}$, have dimensions p_{1n}, p_{2n}, respectively, where $p_{1n} \leq n$ and $p_{1n} + p_{2n} = p_n$.
- Let $\mathbf{X}_{1n}$ and $\mathbf{X}_{2n}$ be the sub-matrices of $\mathbf{X}_n$ corresponding to $\boldsymbol{\beta}_{1n}$ and $\boldsymbol{\beta}_{2n}$, respectively.
- Let us assume true parameter vector

$$\boldsymbol{\beta}_0 = (\beta_{01}, \cdots, \beta_{0p_n})' = (\boldsymbol{\beta}'_{10}, \boldsymbol{\beta}'_{20})'.$$

Shrinkage Estimator for High Dimensional Data

- Let S_{10} and S_{20} represent the corresponding index sets for $\boldsymbol{\beta}_{10}$ and $\boldsymbol{\beta}_{20}$, respectively.
- Specifically, S_{10} includes important predictors and S_{20} includes sparse and weak signals satisfying the following assumption.

(A0) $|\beta_{0j}| = O(n^{-\varsigma})$, for $\forall j \in S_{20}$, where $\varsigma > 1/2$ does not change with n.

- Condition (A0) is considered to be the sparsity of the model. A simpler representation for the finite sample is that $\beta_{0j} = 0 \; \forall j \in S_{20}$, that is, most coefficients are 0 exactly.

A Class of Submodels

- Predictors indexed by S_{10} are used to construct a submodel.
- However, other predictors, especially ones in S_{20} may also make some contributions to the response and cannot be ignored.

Consider

$$\text{UPI or AI}: \quad (\beta'_{20})' = \mathbf{0}_{p_{2n}}.$$

A Candidate Submodel Estimator

We make the following assumptions on the random error and design matrix of the true model:

(A1) The random error ϵ_i's are independent and identically distributed with mean 0 and variance $0 < \sigma^2 < \infty$. Further, $E(\epsilon_i^m) < \infty$, for an even integer m not depending on n.

(A2) $\rho_{1n} > 0$, for all n, the smallest eigenvalue of $\mathbf{C}_{12n}$

Under (A1-A2) and UPI/AE, the submodel estimator (SME) of $\boldsymbol{\beta}_{1n}$ is defined as

$$\hat{\boldsymbol{\beta}}_{1n}^{SM} = (\mathbf{X}_{1n}'\mathbf{X}_{1n})^{-1}\mathbf{X}_{1n}'\mathbf{y}.$$

A Candidate Full Model Estimator

Weighted Ridge Estimation

We estimate an estimator of β_n by minimizing a partial penalized objective function,

$$\hat{\beta}(r_n) = argmin\{\|\mathbf{y} - \mathbf{X}_{1n}\beta_{1n} - \mathbf{X}_{2n}\beta_{2n}\|^2 + r_n\|\beta_{2n}\|^2\}$$

where "$\|\cdot\|$" is the ℓ_2 norm and $r_n > 0$ is a tuning parameter.

Weighted Ridge Estimation

Since $p_n >> n$ and under the sparsity assumption
Define

$$a_n = c_1 n^{-\omega}, \quad 0 < \omega \leq 1/2,\ c_1 > 0.$$

We define a weighted ridge estimator of $\boldsymbol{\beta}_n$ is denoted as

$$\hat{\boldsymbol{\beta}}_n^{WR}(r_n, a_n) = \begin{pmatrix} \hat{\boldsymbol{\beta}}_{1n}^{WR}(r_n) \\ \hat{\boldsymbol{\beta}}_{2n}^{WR}(r_n, a_n) \end{pmatrix}, \text{ where}$$

$$\hat{\boldsymbol{\beta}}_{1n}^{WR}(r_n) = \hat{\beta}_{1n}(r_n)$$

and for $j \notin S_{10}$,

$$\hat{\beta}_j^{\mathrm{WR}}(r_n, a_n) = \begin{cases} \hat{\beta}_j(r_n, a_n), & \hat{\beta}_j(r_n, a_n) > a_n; \\ 0, & \text{otherwise.} \end{cases}$$

Weighted Ridge Estimation

- We call $\hat{\beta}(r_n, a_n)$ as a weighted ridge estimator from two aspects.
- We use a weighted ridge instead of ridge penalty for the HD shrinkage estimation strategy since we do not want to generate some additional biases caused by an additional penalty on β_{1n} if we already have a candidate subset model.
- Here $\hat{\beta}_{1n}^{WR}(r_n)$ changes with r_n and $\hat{\beta}_{2n}^{WR}(r_n, a_n)$ changes with both r_n and a_n.
- For the notation's convenience, we denote the weighted ridge estimators as $\hat{\beta}_{1n}^{WR}$ and $\hat{\beta}_{2n}^{WR}$.

A Candidate HD Shrinkage Estimator

A HD shrinkage estimators (HD-SE) $\hat{\beta}_{1n}^{S}$ is

$$\hat{\beta}_{1n}^{S} = \hat{\beta}_{1n}^{WR} - (h-2)T_n^{-1}(\hat{\beta}_{1n}^{WR} - \hat{\beta}_{1n}^{SM}),$$

$h > 2$ is the number of nonzero elements in $\hat{\beta}_{2n}^{WR}$

$$T_n = (\hat{\beta}_2^{WR})'(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)\hat{\beta}_2^{WR}/\hat{\sigma}^2, \quad (5)$$

$$\mathbf{M}_1 = \mathbf{I}_n - \mathbf{X}_{1n}(\mathbf{X}_{1n}'\mathbf{X}_{1n})^{-1}\mathbf{X}_{1n}'$$

- $\hat{\sigma}^2$ is a consistent estimator of σ^2.
- For example, we can choose $\hat{\sigma}^2 = \sum_{i=1}^{n}(y_i - \mathbf{x}_i'\hat{\beta}^{SM})^2/(n-1)$ under UPI or AI.

A Candidate HD Positive Shrinkage Estimator

A HD positive shrinkage estimator (HD-PSE),

$$\hat{\beta}_{1n}^{PSE} = \hat{\beta}_{1n}^{WR} - ((h-2)T_n^{-1})_1(\hat{\beta}_{1n}^{WR} - \hat{\beta}_{1n}^{SM}),$$

where $(a)_1 = 1$ and a for $a > 1$ and $a \leq 1$, respectively.

Weighted Ridge Estimation

Let $s_n^2 = \sigma^2 \mathbf{d}_n' \Sigma_n^{-1} \mathbf{d}_n$ for any $p_{12n} \times 1$ vector $\mathbf{d}_n$ satisfying $\|\mathbf{d}_n\| \leq 1$.

$$n^{1/2} s_n^{-1} \mathbf{d}_n' (\hat{\beta}_{12n}^{WR} - \beta_{120}) = n^{-1/2} s_n^{-1} \sum_{i=1}^{n} \epsilon_i \mathbf{d}_n' \Sigma_n^{-1} \mathbf{z}_i + o_P(1)$$

$$\xrightarrow{d} N(0, 1).$$

Asymptotic Distributional Risk

Define

$$\Sigma_{n11} = \lim_{n\to\infty} \mathbf{X}'_{1n}\mathbf{X}_{1n}/n, \quad \Sigma_{n22} = \lim_{n\to\infty} \mathbf{X}'_{2n}\mathbf{X}_{2n}/n,$$
$$\Sigma_{n12} = \lim_{n\to\infty} \mathbf{X}'_{1n}\mathbf{X}_{2n}/n, \quad \Sigma_{n21} = \lim_{n\to\infty} \mathbf{X}'_{2n}\mathbf{X}_{1n}/n,$$
$$\Sigma_{n22.1} = \lim_{n\to\infty} n^{-1}\mathbf{X}'_{2n}\mathbf{X}_{2n} - \mathbf{X}'_{2n}\mathbf{X}_{1n}(\mathbf{X}'_{1n}\mathbf{X}_{1n})^{-1}\mathbf{X}'_{1n}\mathbf{X}_{2n}$$
$$\Sigma_{n11.2} = \lim_{n\to\infty} n^{-1}\mathbf{X}'_{1n}\mathbf{X}_{1n} - \mathbf{X}'_{1n}\mathbf{X}_{2n}(\mathbf{X}'_{2n}\mathbf{X}_{2n})^{-1}\mathbf{X}'_{2n}\mathbf{X}_{1n}$$

Asymptotic Distributional Risk

$$K_n : \boldsymbol{\beta}_{20} = n^{-1/2}\boldsymbol{\delta} \quad \text{and} \quad \boldsymbol{\beta}_{30} = \mathbf{0}_{p_{3n}},$$
$$\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots, \delta_{p_{2n}})' \in \Re^{p_{2n}}, \delta_j \quad \text{is fixed.}$$

- Define $\Delta_n = \boldsymbol{\delta}'\boldsymbol{\Sigma}_{n22.1}\boldsymbol{\delta}$,
- $n^{1/2}\mathbf{d}'_{1n}s_{1n}^{-1}(\boldsymbol{\beta}^*_{1n} - \boldsymbol{\beta}_{10})$ is asymptotically normal under $\{K_n\}$, where $s^2_{1n} = \sigma^2\mathbf{d}'_{1n}\boldsymbol{\Sigma}^{-1}_{n11.2}\mathbf{d}_{1n}$.
- The asymptotic distributional risk (ADR) of $\mathbf{d}'_{1n}\boldsymbol{\beta}^*_{1n}$ is

$$\mathrm{ADR}(\mathbf{d}'_{1n}\boldsymbol{\beta}^*_{1n}) = \lim_{n\to\infty} E\{[n^{1/2}s^{-1}_{1n}\mathbf{d}'_{1n}(\boldsymbol{\beta}^*_{1n} - \boldsymbol{\beta}_{10})]^2\}.$$

Mathematical Proof

Under regularity conditions and K_n, and suppose there exists $0 \leq c \leq 1$ such that $c = \lim_{n\to\infty} s_{1n}^{-2}\mathbf{d}'_{1n}\Sigma_{n11}^{-1}\mathbf{d}_{1n}$, we have

$$\mathrm{ADR}(\mathbf{d}'_{1n}\hat{\beta}_{1n}^{WR}) = 1, \tag{6a}$$

$$\mathrm{ADR}(\mathbf{d}'_{1n}\hat{\beta}_{1n}^{SM}) = 1 - (1-c)(1-\Delta_{\mathbf{d}_{1n}}), \tag{6b}$$

$$\mathrm{ADR}(\mathbf{d}'_{1n}\hat{\beta}_{1n}^{S}) = 1 - E[g_1(\mathbf{z}_2+\delta)], \tag{6c}$$

$$\mathrm{ADR}(\mathbf{d}'_{1n}\hat{\beta}_{1n}^{PSE}) = 1 - E[g_2(\mathbf{z}_2+\delta)], \tag{6d}$$

$$\Delta_{\mathbf{d}_{1n}} = \frac{\mathbf{d}'_{1n}(\Sigma_{n11}^{-1}\Sigma_{n12}\delta\delta'\Sigma_{n21}\Sigma_{n11}^{-1})\mathbf{d}_{1n}}{\mathbf{d}'_{1n}(\Sigma_{n11}^{-1}\Sigma_{n12}\Sigma_{n22.1}^{-1}\Sigma_{n21}\Sigma_{n11}^{-1})\mathbf{d}_{1n}}.$$

$$s_{2n}^{-1}\mathbf{d}'_{2n}\mathbf{z}_2 \to N(0,1)$$

$$\mathbf{d}_{2n} = \Sigma_{n21}\Sigma_{n11}^{-1}\mathbf{d}_{1n}$$

$$s_{2n}^2 = \mathbf{d}'_{2n}\Sigma_{n22.1}^{-1}\mathbf{d}_{2n}$$

Asymptotic Distributional Risk Analysis

Mathematical Proof

$$g_1(\mathbf{x}) = \lim_{n\to\infty}(1-c)\frac{p_{2n}-2}{\mathbf{x}'\Sigma_{n22.1}\mathbf{x}}\left[2-\frac{\mathbf{x}'((p_{2n}+2)\mathbf{d}_{2n}\mathbf{d}_{2n}')\mathbf{x}}{s_{2n}^2\mathbf{x}'\Sigma_{n22.1}\mathbf{x}}\right],$$

$$\begin{aligned} g_2(\mathbf{x}) = \quad & \lim_{n\to\infty}\frac{p_{2n}-2}{\mathbf{x}'\Sigma_{n22.1}\mathbf{x}}\left[(1-c)\left(2-\frac{\mathbf{x}'((p_{2n}+2)\mathbf{d}_{2n}\mathbf{d}_{2n}')\mathbf{x}}{s_{2n}^2\mathbf{x}'\Sigma_{n22.1}\mathbf{x}}\right)\right] \\ & I(\mathbf{x}'\Sigma_{n22.1}\mathbf{x} \geq p_{2n}-2) \\ & +\lim_{n\to\infty}[(2-s_{2n}^{-2}\mathbf{x}'\delta_{2n}\delta_{2n}'\mathbf{x})(1-c)]I(\mathbf{x}'\Sigma_{n22.1}\mathbf{x} \leq p_{2n}-2) \end{aligned}$$

Moral of the Story

By Ignoring the Bias, it will Not go away!

- Submodel estimator provided by some existing variable selection techniques when $p_n \gg n$ are subject to bias.
- The prediction performance can be improved by the shrinkage strategy.
- Particulary when an under-fitted submodel is selected by an aggressive penalty parameter.

Moral of the Story

By Ignoring the Bias, it will Not go away!

- When $p \gg n$, we assume the true model is sparse in the sense that most coefficients goes to 0 when $n \to \infty$.
- However, it is realistic to assume that some β_j may be small, but not exactly 0.
- Such predictors with small amount of influence on the response variable are often ignored incorrectly in HD variable selection methods.
- We borrow (re-gain) some information from those predictors using the shrinkage strategy to improve the prediction performance.

Engineering Proof: Simulation

- In all experiments, ϵ_i's are simulated from i.i.d standard normal random variables, $x_{is} = (\xi^1_{(is)})^2 + \xi^2_{(is)}$, where $\xi^1_{(is)}$ and $\xi^2_{(is)}$, $i = 1, \cdots, n$, $s = 1, \cdots, p_n$ are also independent copies of standard normal distribution.
- In all sampling experiments, we let $p_n = n^\alpha$ for different sample size n, where α changes from 1 to 1.8 with an increment of 0.2. The HD-PSE is computed for $r_n = p_n^{1/8}$ and $a_n = 0.1n^{-1/3}$.

Engineering Proof

- The performance of an estimator of β will be appraised using the mean squared error (MSE) criterion.
- All computations were conducted using the **R** statistical software.
- We have numerically calculated the relative MSE of the estimators with respect to $\hat{\beta}^{WR}$ by simulation.
- The simulated relative efficiency (SRE) of the estimator $\beta^{\diamond}$ to the maximum likelihood estimator $\hat{\beta}^{FM}$ is denoted by

$$\text{SRE}(\hat{\beta}^{FM} : \beta^{\diamond}) = \frac{\text{MSE}(\hat{\beta}^{WR})}{\text{MSE}(\beta^{\diamond})}.$$

- A SRE larger than one indicates the degree of superiority of the estimator $\beta^{\diamond}$ over $\hat{\beta}^{WR}$.

Simulation Results

Engineering Proof

Relative Performance

- We let $\beta_{10} = (1.5, 3, 2)'$ be fixed for every design.
- Let $\Delta^* = \|\beta_{20} - \mathbf{0}\|^2$ varying between 0 and 4.
- We choose $n = 30$ or 100.

Table: Simulated RMSEs

.

(n,p)	Δ^*	$\hat{\beta}_{1n}^{SM}$	$\hat{\beta}_{1n}^{PSE}$	(n,p)	Δ^*	$\hat{\beta}_{1n}^{SM}$	$\hat{\beta}_{1n}^{PSE}$
	0.00	16.654	4.101		0.00	8.953	5.385
	0.05	8.202	3.446		0.05	4.456	3.794
	0.20	2.855	2.610		0.20	1.551	3.216
	0.25	2.074	2.437		0.25	1.422	2.833
	0.30	1.857	2.180		0.30	1.091	2.459
$(30,30)$	0.35	1.643	1.949	$(30,59)$	0.35	0.986	2.447
	0.80	0.649	1.506		0.80	0.542	1.601
	2.50	0.232	1.160		2.50	0.234	1.171
	3.30	0.170	1.095		3.30	0.210	1.108
	0.00	12.672	4.260		0.00	5.546	5.388
	0.05	2.546	3.538		0.05	1.255	1.900
	0.10	1.129	3.256		0.15	0.441	1.322
	0.20	0.628	2.948		0.20	0.361	1.382
	0.25	0.481	3.366		0.25	0.316	1.358
$(100,158)$	0.40	0.311	2.272	$(100,398)$	0.40	0.198	1.543
	1.40	0.110	1.500		1.40	0.096	1.826
	3.10	0.066	1.181		3.10	0.079	1.304
	3.50	0.060	1.217		3.50	0.075	1.297

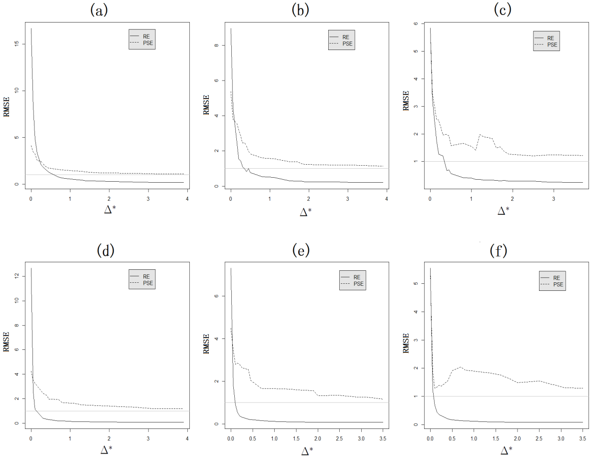


Figure: The top three panels (a-c) are for $n = 30$ and $p_n = 30, 59, 117$ from the left to the right. The bottom panels (d-f) are for $n = 100$ and $p_n = 158, 251, 398$ from the left to the right. Solid curves: $\mathrm{RMSE}(\hat{\beta}_{1n}^{SM})$; Dashed curves: $\mathrm{RMSE}(\hat{\beta}_{1n}^{PSE})$.

Shrinkage Versus Penalty Estimators

Engineering Solution: Simulation Results

- Performance of HD-PSE relative to penalty estimators including Lasso, ALasso, SCAD, MCP and Threshold Ridge (TR).
- We let $\beta_{10} = (1.5, 3, 2, \underbrace{0.1, \cdots, 0.1}_{p_{1n}-3})'$, $\beta_{20} = \mathbf{0}'_{p_{2n}}$.
- The model includes some predictors with weak signals. We consider $n = 30$ and $p_{1n} = 3, 4, 10, 20$.
- We choose $a = 3.7$ and $\gamma = 3$ for SCAD and MCP, respectively.
- For TR, we choose $\alpha_n = c_6 n^{-1/3}$ and $\lambda = c_7(\log\log n)^3/\alpha_n^2$, where c_6 and c_7 are two tuning parameters.
- All tuning parameters are chosen using the generalized cross validation.

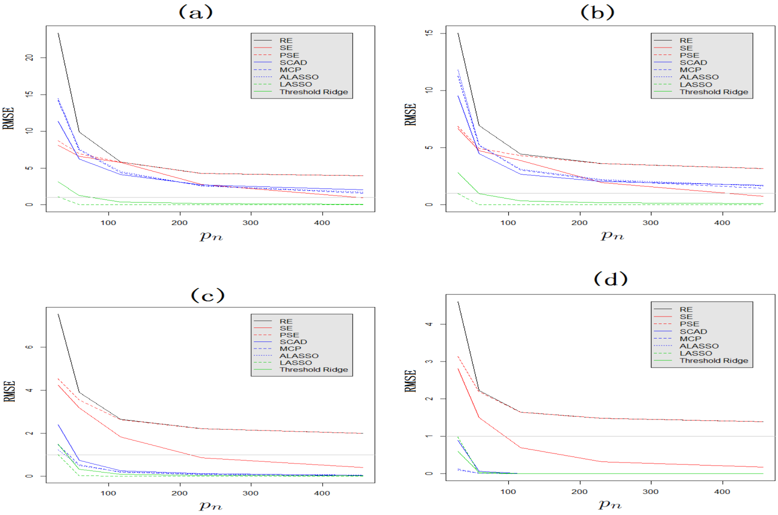
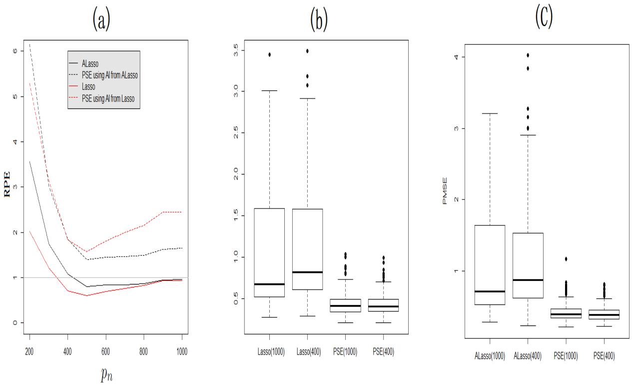


Figure: RMSEs for $n = 30$. Plots (a-d) are for $p_1 = 3, 4, 10, 20$, respectively.



(a)
(b)
(C)
RPE
ALasso
PSE using AI from ALasso
Lasso
PSE using AI from Lasso
p_n
Lasso(1000)
Lasso(400)
PSE(1000)
PSE(400)
PMSE
ALasso(1000)
ALasso(400)

p_1	p_n	$\hat{\beta}_{1n}^{SM}$	$\hat{\beta}_{1n}^{PSE}$	$\hat{\beta}_{1n}^{\mathrm{SCAD}}$	$\hat{\beta}_{1n}^{\mathrm{MCP}}$	$\hat{\beta}_{1n}^{\mathrm{ALasso}}$	$\hat{\beta}_{1n}^{\mathrm{Lasso}}$	$\hat{\beta}_{1n}^{\mathrm{TR}}$
3	30	23.420	8.740	14.486	14.247	11.399	3.130	1.097
	59	9.900	6.951	7.588	7.499	6.244	1.257	0.015
	231	4.292	4.291	2.568	2.622	2.714	0.166	0.003
	456	3.977	3.977	1.739	1.576	2.059	0.099	0.002
4	30	15.055	6.882	11.809	11.291	9.528	2.830	0.993
	59	6.954	4.933	5.260	5.204	4.469	0.966	0.019
	231	3.605	3.605	2.222	2.154	2.045	0.167	0.004
	456	3.184	3.184	1.648	1.436	1.703	0.102	0.003
10	30	7.528	4.526	1.232	1.469	2.391	1.497	1.001
	59	3.899	3.534	0.493	0.538	0.746	0.321	0.032
	231	2.212	2.212	0.104	0.083	0.117	0.034	0.005
	456	1.997	1.997	0.052	0.032	0.050	0.017	0.003
20	30	4.603	3.139	0.099	0.128	0.892	0.599	0.981
	59	2.231	2.194	0.016	0.018	0.067	0.031	0.013
	231	1.489	1.489	0.002	0.002	0.003	0.002	0.002
	456	1.392	1.392	0.001	0.001	0.002	0.001	0.001

Threshold Ridge Regression

A Threshold ridge (TR) for $1 \leq j \leq p_n$ of β_j is given by (Shao and Deng (2008))

$$\hat{\beta}_j^{\mathrm{TR}} = \begin{cases} \widetilde{\beta}_j, & |\widetilde{\beta}_j| > a_n, \\ 0, & |\widetilde{\beta}_j| \leq a_n, \end{cases}$$

where

$$\widetilde{\beta}_n = \arg\min_{\beta} \left\{ \sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{p_n} x_{ij}\beta_j \right)^2 + \lambda \sum_{j=1}^{p_n} \beta_j^2 \right\}$$

and $a_n = cn^{-\omega}$ for $0 < \omega < 1/2$ and $c > 0$.

Shrinkage Versus Penalty Estimators

- The submodel estimator dominates all other estimators in the class, since $\hat{\beta}^{SM}$ is computed based on the true submodel.
- SCAD and MCP work better than the HD-PSE for smaller p_n.
- HD-PSE performs better than penalty estimators for larger p_n.
- Penalty estimators are even less efficient than the weighted ridge estimate.
- This phenomenon can be explained by the existence of predictors with weak effects, which cannot be separated from zero effects using Lasso-type methods.
- The predictors are designed to be correlated, the weighted ridge step can generate a better estimation at the starting point.

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- SCAD and MCP work better than the HD-PSE for smaller p_n.
- HD-PSE performs better than penalty estimators for larger p_n.
- Penalty estimators are even less efficient than the weighted ridge estimate.
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Microarray Data Example

- We apply the proposed HD-PSE strategy to the data set reported in Scheetz et al. (2006) and also analyzed by Huang, Ma and Zhang (2008).
- In this dataset, 120 twelve-week-old male offsprings of F1 animals were selected for tissue harvesting from the eyes for microarray analysis.
- The microarrays used to analyze the RNA from the eyes of these F2 animals contain over $31,042$ different probe sets (Affymetric GeneChip Rat Genome 230 2.0 Array).

Microarray Data Example

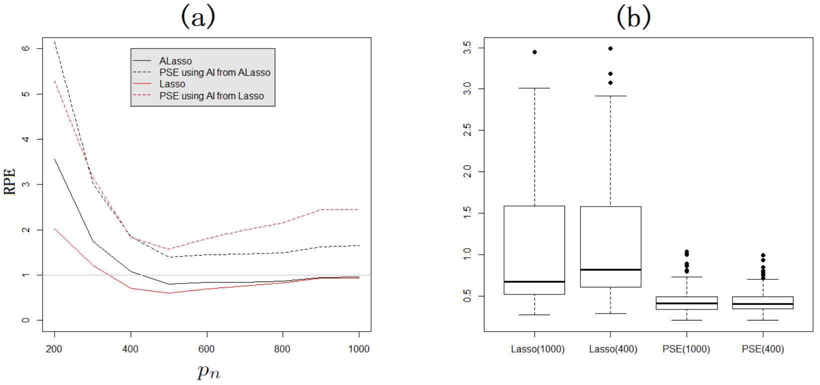
- Huang, Ma and Zhang (2008) studied a total of 18,976 probes including gene TRIM32, which was recently found to cause Bardet-Biedl syndrome (Chiang et al. (2006)), a genetically heterogeneous disease of multiple organ systems including the retina.
- A regression analysis was conducted to find the probes among the remaining $18,975$ probes that are most related to TRIM32 (Probe ID: 1389163_at). Huang et al (2008) found 19 and 24 probes based on Lasso and adaptive Lasso methods, respectively.
- We compute HD-PSEs based on two different candidate subset models consisting of 24 and 19 probes selected from Lasso and adaptive Lasso, respectively.

Microarray Data Example

- In the largest full set model, we consider at most $1,000$ probes with the largest variances. Other smaller full set model with top p_n probes are also considered.
- Here we choose different p_n's between 200 and $1,000$.
- The relative prediction error (RPE) of the estimator $\beta^*_{\mathcal{J}}$ relative to weighted ridge estimator $\hat{\beta}^{WR}_{\mathcal{J}}$ is computed as follows

$$\mathrm{RPE}(\beta^*_{\mathcal{J}}) = \frac{\sum_{i=1}^{n} \|\mathbf{y} - \sum_{j\in\mathcal{J}} \mathbf{X}_{\mathcal{J}} \hat{\beta}^{WR}_{\mathcal{J}}\|^2}{\sum_{i=1}^{n} \|\mathbf{y} - \sum_{j\in\mathcal{J}} \mathbf{X}_{\mathcal{J}} \beta^*_{\mathcal{J}}\|^2},$$

where $\mathcal{J}$ is the index of the submodel including either 24 or 19 elements.


(a)
RPE
ALasso
PSE using AI from ALasso
Lasso
PSE using AI from Lasso
0
1
2
3
4
5
6
200
400
600
800
1000
p_n
(b)
0.5
1.0
1.5
2.0
2.5
3.0
3.5
Lasso(1000)
Lasso(400)
PSE(1000)
PSE(400)

Envoi

- We generalized the classical Stein's shrinkage estimation to a high-dimensional sparse model with some predictors with weak signals.
- When p_n grows with n quickly, it is reasonable to suspect that most predictors do not contribute, that is model is sparse.
- We proposed a HD shrinkage estimation strategy by shrinking a weighted ridge estimator in the direction of a candidate submodel.

Envoi

- Existing penalized regularization approaches have some advantages of generating a parsimony sparse model, but tends to ignore the possible small contributions from some predictors.
- Lasso-type methods provide estimation and prediction only based on the selected candidate submodel, which is often inefficient with the existence of mild or weak signals.
- Our proposed HD shrinkage strategy takes into account possible contributions of all other possible nuisance parameters and has dominant prediction performances over submodel estimates generated from Lasso-type methods, which depend strongly on the sparsity assumption of the true model.

Envoi

- Gauss offered two justifications for least squares: First, what we now call the maximum likelihood argument in the Gaussian error model. Second, the concept of risk and the start of what we now call the Gauss-Markov theorem.
- Stein's 1956 paper revealed that neither maximum likelihood estimators nor unbiased estimators have desirable risk functions when the dimension of the parameter space is not small.
- PSE outperforms the maximum likelihood estimator of the regression parameter vector in the entire parameter space.

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Envoi

- Big data is the future of Science and Transdisciplinary research in Statistical Sciences is a must.
- A greater collaboration between statisticians, computer scientists and social scientists (Facebook clicks, Netflix queues, and GPS data, a few to mention)
- Data is never neutral and unbiased, we must pull expertise across a host of fields to combat the biases in the estimation.

Is Classical Shrinkage Estimation Dead?

Long Live L_2 Shrinkage!

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- Study classical problems - Classical assumptions
- Exact/Analytic Solutions
- Low-dimensional Data Analysis
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- Glory of the Individual

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Thank you!

Thank you and thanks to organizers!