Happy Bastille Day!

# Elliott's program and descriptive set theory III

Ilijas Farah
(joint work with Bradd Hart and David Sherman
and with

# Elliott's program and descriptive set theory III

Ilijas Farah
(joint work with Bradd Hart and David Sherman
and with
orge Elliott, Vern Paulsen, Christian Rosendal, Andrew Toms and Asger Törnqui
LC 2012, Manchester, July 14

model theory

# Elliott's program and ~~descriptive set theory~~ III

Ilijas Farah
(joint work with Bradd Hart and David Sherman
and with
orge Elliott, Vern Paulsen, Christian Rosendal, Andrew Toms and Asger Törnqui
LC 2012, Manchester, July 14

model theory

# Elliott's program and ~~descriptive set theory~~ III

Ilijas Farah
(joint work with Bradd Hart and David Sherman
and with
orge Elliott, Vern Paulsen, Christian Rosendal, Andrew Toms and Asger Törnqui

LC 2012, Manchester, July 14

*As logicians, we do our subject a disservice by convincing others that logic is first order, and then convincing them that almost none of the concepts of modern mathematics can really be captured in first order logic. (Jon Barwise)*

# The plan

1. Thursday:
   1.1 Basic properties of C*-algebras.
   1.2 Classification: UHF and AF algebras.
   1.3 Elliott's program.
2. Yesterday: Applying logic to 1.2–1.3.
   2.1 Set theory.
   2.2 C*-algebras (review).
   2.3 More set theory.
3. Today: Convincing you that 1.2–1.3 is logic.

# The plan

1. Thursday:
   1.1 Basic properties of C*-algebras.
   1.2 Classification: UHF and AF algebras.
   1.3 Elliott's program.
2. Yesterday: Applying logic to 1.2–1.3.
   2.1 Set theory.
   2.2 C*-algebras (review).
   2.3 More set theory.
3. Today: Convincing you that 1.2–1.3 is logic.
   3.1 Review.

# The plan

1. Thursday:
   1.1 Basic properties of C*-algebras.
   1.2 Classification: UHF and AF algebras.
   1.3 Elliott's program.
2. Yesterday: Applying logic to 1.2–1.3.
   2.1 Set theory.
   2.2 C*-algebras (review).
   2.3 More set theory.
3. Today: Convincing you that 1.2–1.3 is logic.
   3.1 Review.
   3.2 Logic of metric structures.

# The plan

1. Thursday:
   1.1 Basic properties of C*-algebras.
   1.2 Classification: UHF and AF algebras.
   1.3 Elliott's program.
2. Yesterday: Applying logic to 1.2–1.3.
   2.1 Set theory.
   2.2 C*-algebras (review).
   2.3 More set theory.
3. Today: Convincing you that 1.2–1.3 is logic.
   3.1 Review.
   3.2 Logic of metric structures.
   3.3 A proof from the book.

# The plan

1. Thursday:
   1.1 Basic properties of C*-algebras.
   1.2 Classification: UHF and AF algebras.
   1.3 Elliott's program.
2. Yesterday: Applying logic to 1.2–1.3.
   2.1 Set theory.
   2.2 C*-algebras (review).
   2.3 More set theory.
3. Today: Convincing you that 1.2–1.3 is logic.
   3.1 Review.
   3.2 Logic of metric structures.
   3.3 A proof from the book.
   3.4 It is all logic.

# Review I: C*-algebras

C*-algebras are norm-closed subalgebras of $\mathcal{B}(H)$, the algebra of bounded linear operators on a complex Hilbert space $H$.

# Review I: C*-algebras

C*-algebras are norm-closed subalgebras of $\mathcal{B}(H)$, the algebra of bounded linear operators on a complex Hilbert space $H$.
Separable unital algebras that are direct limits of finite-dimensional C*-algebras (UHF and AF algebras) were classified by Glimm and Elliott.

# Review I: C*-algebras

C*-algebras are norm-closed subalgebras of $\mathcal{B}(H)$, the algebra of bounded linear operators on a complex Hilbert space $H$.

Separable unital algebras that are direct limits of finite-dimensional C*-algebras (UHF and AF algebras) were classified by Glimm and Elliott.

Elliott's program: Classify separable, unital, simple, nuclear C*-algebra by K-theoretic invariants.

# Review I: C*-algebras

C*-algebras are norm-closed subalgebras of $\mathcal{B}(H)$, the algebra of bounded linear operators on a complex Hilbert space $H$.

Separable unital algebras that are direct limits of finite-dimensional C*-algebras (UHF and AF algebras) were classified by Glimm and Elliott.

Elliott's program: Classify separable, unital, simple, nuclear C*-algebra by K-theoretic invariants.

Are there set-theoretic obstructions to this?

# Review II: Borel reductions

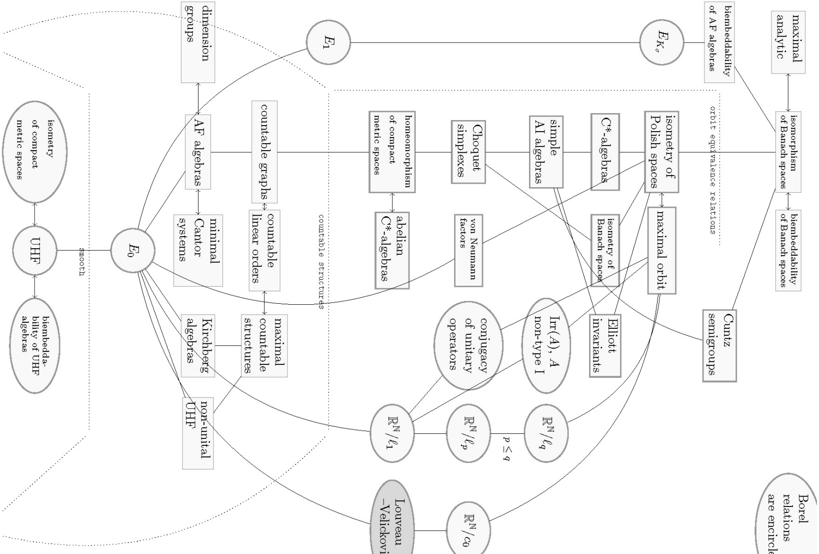
$E \leq_B F$ iff there exists a Borel function $f$ such that

$$x \, E \, y \text{ iff } f(x) \, F \, f(y).$$

# Review II: Borel reductions

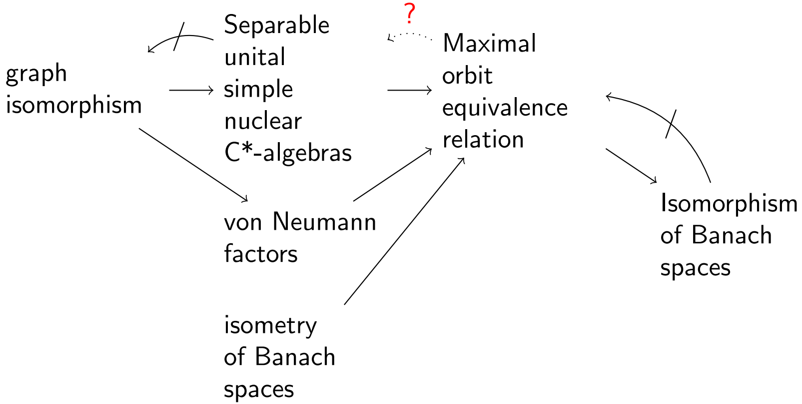
$E \leq_B F$ iff there exists a Borel function $f$ such that

$$x \, E \, y \text{ iff } f(x) \, F \, f(y).$$



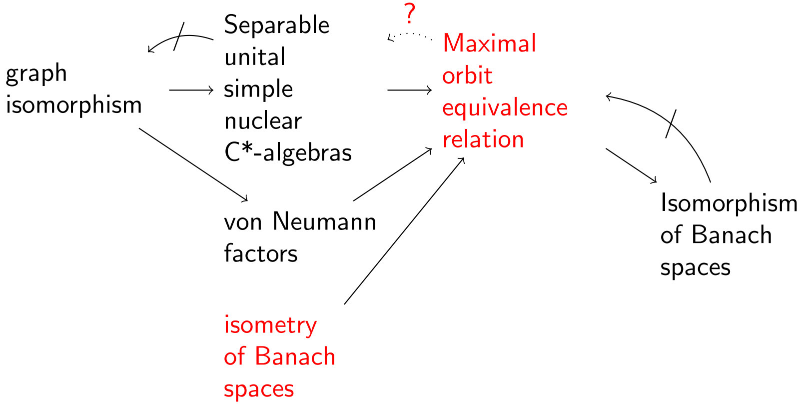
# Review II: Set theory

By results of F.–Toms–Törnquist, Ferenczi–Louveau–Rosendal and Melleray:



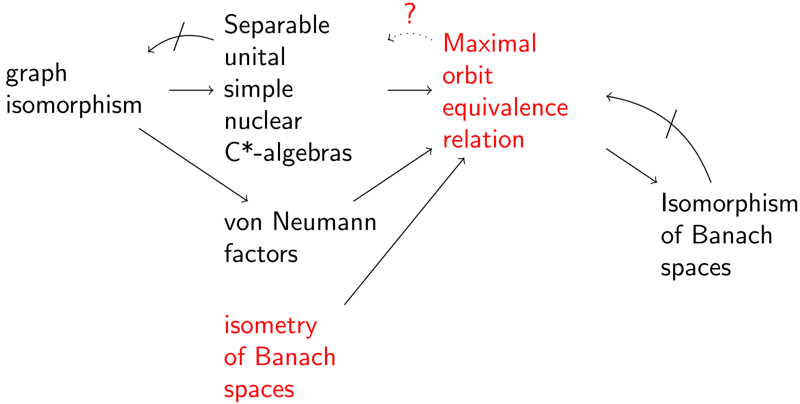
# Review II: Set theory

By results of F.–Toms–Törnquist, Ferenczi–Louveau–Rosendal and Melleray:



# Review II: Set theory

By results of F.–Toms–Törnquist, Ferenczi–Louveau–Rosendal and Melleray:




## Question

*Is the isomorphism of separable C*-algebras $\leq_B$ an orbit equivalence relation of a Polish group action?*

# Review: Urysohn space, $\mathbb{U}$

It is a separable complete metric space which is universal for separable metric spaces and such that for all finite metric $X \subseteq Y$, every isometry $f : X \to \mathbb{U}$ extends to an isometry $g : Y \to \mathbb{U}$.

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{U} \\ Y \arrow[u, "\subseteq"] \arrow[ur, "g"'] & \end{tikzcd}$$

# Review: Urysohn space, $\mathbb{U}$

It is a separable complete metric space which is universal for separable metric spaces and such that for all finite metric $X \subseteq Y$, every isometry $f\colon X \to \mathbb{U}$ extends to an isometry $g\colon Y \to \mathbb{U}$.

$$\begin{tikzcd}
X \arrow[r, "f"] & \mathbb{U} \\
Y \arrow[u, "\subseteq"] \arrow[ur, "g"'] &
\end{tikzcd}$$

**Theorem (Clemens–Gao–Kechris, 2000)**

*The orbit equivalence relation of* $\mathrm{Iso}(\mathbb{U}) \curvearrowright F(\mathbb{U})$ *is the* $\leq_B$*-maximal among orbit equivalence relations of Polish group actions.*

# Logic of metric structures

Developed by C.W. Henson, I. Ben Ya'acov, A. Berenstein, and A. Usvyatsov.

[1] several times

# Logic of metric structures

Developed by C.W. Henson, I. Ben Ya'acov, A. Berenstein, and A. Usvyatsov.
I shall describe only the 'logic of C*-algebras' as modified[1] by F.–Hart–Sherman.

[1] several times

# Logic of C*-algebras: Syntax

Language: $\{+, \cdot, ^*\}$.

Terms $(s, t, \dots)$:

noncommutative *-polynomials.

# Logic of C*-algebras: Syntax

Language: $\{+, \cdot, ^*\}$.

Terms $(s, t, \dots)$:

noncommutative *-polynomials.

Atomic formulas $(\varphi, \psi, \dots)$:

$\|t\|$ for a term $t$.

# Logic of C*-algebras: Syntax

Language: $\{+, \cdot, ^*\}$.

Terms $(s, t, \dots)$:

noncommutative *-polynomials.

Atomic formulas $(\varphi, \psi, \dots)$:

$\|t\|$ for a term $t$.

Formulas $(\varphi, \psi, \dots)$:

The smallest set $\mathbb{F}$ that satisfies

1. all atomic formulas are in $\mathbb{F}$,

# Logic of C*-algebras: Syntax

Language: $\{+, \cdot, ^*\}$.

Terms $(s, t, \dots)$:

noncommutative *-polynomials.

Atomic formulas $(\varphi, \psi, \dots)$:

$\|t\|$ for a term $t$.

Formulas $(\varphi, \psi, \dots)$:

The smallest set $\mathbb{F}$ that satisfies

1. all atomic formulas are in $\mathbb{F}$,
2. if $g\colon \mathbb{R}^n \mapsto \mathbb{R}$ is uniformly continuous and $\varphi_1, \dots, \varphi_n$ are in $\mathbb{F}$ then

$$g(\varphi_1, \dots, \varphi_n)$$

is in $\mathbb{F}$,

# Logic of C*-algebras: Syntax

Language: $\{+, \cdot, {}^*\}$.

Terms $(s, t, \dots)$:

noncommutative *-polynomials.

Atomic formulas $(\varphi, \psi, \dots)$:

$\|t\|$ for a term $t$.

Formulas $(\varphi, \psi, \dots)$:

The smallest set $\mathbb{F}$ that satisfies

1. all atomic formulas are in $\mathbb{F}$,
2. if $g\colon \mathbb{R}^n \mapsto \mathbb{R}$ is uniformly continuous and $\varphi_1, \dots, \varphi_n$ are in $\mathbb{F}$ then
$$g(\varphi_1, \dots, \varphi_n)$$
is in $\mathbb{F}$,
3. $\sup_{\|x_i\| \leq 1} \varphi$ and $\inf_{\|x_i\| \leq 1} \varphi$ are in $\mathbb{F}$ whenever $\varphi$ is in $\mathbb{F}$.

# Logic of C*-algebras: Semantics

If $A$ is a normed metric structure with operations $+, \cdot, ^*$ that are uniformly continuous on bounded sets and $\varphi(x)$ is a formula then

$$\varphi(x)^A$$

is interpreted in the natural way.

Its interpretation is a function into $\mathbb{R}$ that is uniformly continuous on bounded sets.

# Logic of C*-algebras: Semantics

If $A$ is a normed metric structure with operations $+, \cdot, ^*$ that are uniformly continuous on bounded sets and $\varphi(x)$ is a formula then

$$\varphi(x)^A$$

is interpreted in the natural way.

Its interpretation is a function into $\mathbb{R}$ that is uniformly continuous on bounded sets.

# Logic of C*-algebras: Semantics

If $A$ is a normed metric structure with operations $+, \cdot, ^*$ that are uniformly continuous on bounded sets and $\varphi(x)$ is a formula then

$$\varphi(x)^A$$

is interpreted in the natural way.
Its interpretation is a function into $\mathbb{R}$ that is uniformly continuous on bounded sets.

## Example

Fix C*-algebra $A$.

1. If $\varphi_P(x)$ is $\|x^2 - x\| + \|x - x^*\|$ then the zero-set of $\varphi_P$

   $$\{a \in A | \varphi_P(a)^A = 0\}$$

   is the set of projections in $A$.

# Logic of C*-algebras: Semantics

If $A$ is a normed metric structure with operations $+, \cdot, ^*$ that are uniformly continuous on bounded sets and $\varphi(x)$ is a formula then

$$\varphi(x)^A$$

is interpreted in the natural way.
Its interpretation is a function into $\mathbb{R}$ that is uniformly continuous on bounded sets.

Example

Fix C*-algebra $A$.

1. If $\varphi_P(x)$ is $\|x^2 - x\| + \|x - x^*\|$ then the zero-set of $\varphi_P$

$$\{a \in A | \varphi_P(a)^A = 0\}$$

   is the set of projections in $A$.
2. If $\psi_{MvN}(x, y) = \varphi_P(x) + \varphi_P(y) + \inf_{\|z\| \leq 1}(\|x - zz^*\| + \|y - z^*z\|)$, then the zero set of $\psi_{MvN}$ is $\{(p, q) : p \sim q\}$.

# Theory of a C*-algebra $A$, $\mathrm{Th}(A)$

A theory of a C*-algebra $A$ is the set

$$\{\varphi : \varphi^A = 0\}.$$

Alternatively, one could define the theory of $A$ as the map from the set of all sentences into $\mathbb{R}^+$:

$$\varphi \mapsto \varphi^A.$$

# Theory of a C*-algebra $A$, $\mathrm{Th}(A)$

A theory of a C*-algebra $A$ is the set

$$\{\varphi : \varphi^A = 0\}.$$

Alternatively, one could define the theory of $A$ as the map from the set of all sentences into $\mathbb{R}^+$:

$$\varphi \mapsto \varphi^A.$$

With any natural Borel space of models and Borel space of formulas, one has the following

**Lemma**

*The map $A \mapsto \mathrm{Th}(A)$ is Borel.*

# A short intermission

*Your theorem is not as good as you think when you prove it*
*(Gert K. Pedersen)*

# A short intermission

*Your theorem is not as good as you think when you prove it and it is not as bad as you think five days later.*
*(Gert K. Pedersen)*

# A short intermission

*Your theorem is not as good as you think when you prove it and it is not as bad as you think five days later.*
*(Gert K. Pedersen)*

*He [G.K. Pedersen] was obsessed with being witty.*
*(Anonymous)*

# Borel space of separable (complete) models

Fix a language $\mathcal{L} = \{f_1, f_2, \dots\}$ in the logic of metric structures and a $\mathcal{L}$-theory $T$.

# Borel space of separable (complete) models

Fix a language $\mathcal{L} = \{f_1, f_2, \dots\}$ in the logic of metric structures and a $\mathcal{L}$-theory $T$.
Let

$$\mathbb{X} = (X, d, F_1^X, F_2^X, \dots)$$

be a model of $T$, where $(X, d)$ is separable, complete metric space and $F_j^X \subseteq X^{k(j)}$ is a closed set corresponding to the graph of the interpretation of $f_j$.

# Borel space of separable (complete) models II

By a result of Katětov, $X$ can be extended to a complete, separable metric space $X^+$ isometric to $\mathbb{U}$. Moreover, any isometry between $X$ and $Y$ can be extended to an isometry between $X^+$ and $Y^+$.

# Borel space of separable (complete) models II

By a result of Katětov, $X$ can be extended to a complete, separable metric space $X^+$ isometric to $\mathbb{U}$. Moreover, any isometry between $X$ and $Y$ can be extended to an isometry between $X^+$ and $Y^+$. Identify $X^+$ with $\mathbb{U}$ to get

$$\mathbb{X}^+ = (\mathbb{U}, X, d, F_1^X, F_2^X, \dots).$$

# Borel space of separable (complete) models II

By a result of Katětov, $X$ can be extended to a complete, separable metric space $X^+$ isometric to $\mathbb{U}$. Moreover, any isometry between $X$ and $Y$ can be extended to an isometry between $X^+$ and $Y^+$. Identify $X^+$ with $\mathbb{U}$ to get

$$\mathbb{X}^+ = (\mathbb{U}, X, d, F_1^X, F_2^X, \dots).$$

The space of such $\mathbb{X}^+$ carries a standard Borel structure, and the action of $\mathrm{Iso}(\mathbb{U})$ by translations is Borel.

# Borel space of separable (complete) models II

By a result of Katětov, $X$ can be extended to a complete, separable metric space $X^+$ isometric to $\mathbb{U}$. Moreover, any isometry between $X$ and $Y$ can be extended to an isometry between $X^+$ and $Y^+$. Identify $X^+$ with $\mathbb{U}$ to get

$$\mathbb{X}^+ = (\mathbb{U}, X, d, F_1^X, F_2^X, \dots).$$

The space of such $\mathbb{X}^+$ carries a standard Borel structure, and the action of $\mathrm{Iso}(\mathbb{U})$ by translations is Borel.
Also

$$(X, F_1^X, \dots) \cong (Y, F_1^Y, \dots)$$

iff $\mathbb{X}$ and $\mathbb{Y}$ are in the same $\mathrm{Iso}(\mathbb{U})$-orbit.

Theorem (Elliott–F.–Paulsen–Rosendal–Toms–Törnquist)

*Assume $\mathcal{L}$ is a countable language in the logic of metric structures, and $T$ is a $\mathcal{L}$-theory. Then the isomorphism of separable (complete) models of $T$ is $\leq_B$ an orbit equivalence relation of an action of* $\mathrm{Iso}(\mathbb{U})$*, the isometry group of Urysohn's space.*

Theorem (Elliott–F.–Paulsen–Rosendal–Toms–Törnquist)

*Assume $\mathcal{L}$ is a countable language in the logic of metric structures, and $T$ is a $\mathcal{L}$-theory. Then the isomorphism of separable (complete) models of $T$ is $\leq_B$ an orbit equivalence relation of an action of $\mathrm{Iso}(\mathbb{U})$, the isometry group of Urysohn's space.*

Corollary (Elliott–F.–Paulsen–Rosendal–Toms–Törnquist)

*The isomorphism relation of separable C*-algebras is $\leq_B$ an orbit equivalence relation of a Polish group action.*

# A perfect analogy

| | | |
|---|---|---|
| countable models | $\longleftrightarrow$ | Actions of $S_\infty$ |
| complete separable metric models | $\longleftrightarrow$ | Actions of $\mathrm{Iso}(\mathbb{U})$ |

# A perfect analogy

countable models $\longleftrightarrow$ Actions of $S_\infty$

complete separable metric models $\longleftrightarrow$ Actions of $\mathrm{Iso}(\mathbb{U})$

**Problem**

*Develop a method for distinguishing orbit equivalence relations of turbulent actions of different Polish groups.*

# The definition of nuclear C*-algebras, finally

There are several equivalent ways to define nuclear algebras. I will use one that is most convenient for my purposes.

# The definition of nuclear C*-algebras, finally

There are several equivalent ways to define nuclear algebras. I will use one that is most convenient for my purposes.
It will take some time to define it.

# Positivity

An element $a$ of a C*-algebra is *positive* if $a = b^* b$ for some $b$.

# Positivity

An element $a$ of a C*-algebra is *positive* if $a = b^*b$ for some $b$.
A linear map $\Phi\colon A \to B$ is *positive* if it sends positive elemets to positive elements.

# Positivity

An element $a$ of a C*-algebra is *positive* if $a = b^* b$ for some $b$.
A linear map $\Phi\colon A \to B$ is *positive* if it sends positive elemets to positive elements.
It is *completely positive* if

$$M_n(A) \ni (a_{ij})_{i,j \le n} \mapsto (\Phi(a_{ij}))_{i,j \le n} \in M_n(B)$$

is positive for all $n$.

# Positivity

An element $a$ of a C*-algebra is *positive* if $a = b^*b$ for some $b$.
A linear map $\Phi\colon A \to B$ is *positive* if it sends positive elemets to positive elements.
It is *completely positive* if

$$M_n(A) \ni (a_{ij})_{i,j\leq n} \mapsto (\Phi(a_{ij}))_{i,j\leq n} \in M_n(B)$$

is positive for all $n$.

## Example

1. Every *-homomorphism is completely positive.
2. The transpose map on $M_2(\mathbb{C})$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

is positive but not completely positive.

# Positivity II

### Proposition

*If* $\Phi \colon A \to B$ *is a *-homomorphism and* $p \in B$ *is a projection, then*

$$a \mapsto p\Phi(a)p$$

*is completely positive.*

# Positivity II

**Proposition**

*If $\Phi\colon A \to B$ is a *-homomorphism and $p \in B$ is a projection, then*

$$a \mapsto p\Phi(a)p$$

*is completely positive.*

ucp:= unital completely positive

# Positivity II

**Proposition**

*If* $\Phi \colon A \to B$ *is a *-homomorphism and* $p \in B$ *is a projection, then*

$$a \mapsto p\Phi(a)p$$

*is completely positive.*

ucp:= unital completely positive

ucp maps $\varphi \colon A \to \mathbb{C}$ (aka *states*) play a key role in the GNS theorem.

# Completely Positive Approximation Property (CPAP)

## Definition

A unital C*-algebra $A$ is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps $\varphi_j$ and $\psi_j$ for $j \in \mathbb{N}$

$$A \xrightarrow{\varphi_j} M_{n(k)}(\mathbb{C}) \xrightarrow{\psi_j} A$$

such that $\psi_j \circ \varphi_j$ converges to $\mathrm{id}_A$ pointwise.

# Completely Positive Approximation Property (CPAP)

## Definition

A unital C*-algebra $A$ is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps $\varphi_j$ and $\psi_j$ for $j \in \mathbb{N}$

$$A \xrightarrow{\varphi_j} M_{n(k)}(\mathbb{C}) \xrightarrow{\psi_j} A$$

such that $\psi_j \circ \varphi_j$ converges to $\mathrm{id}_A$ pointwise.

## Lemma

1. *Each $M_n(\mathbb{C})$ is nuclear.*
2. *Direct limits of nuclear algebras are nuclear.*
3. *UHF $\Rightarrow$ AF $\Rightarrow$ nuclear.*
4. *abelian $\Rightarrow$ nuclear.*
5. *$A$ nuclear, $X$ cpct Hausdroff $\Rightarrow C(X, A)$ nuclear.*

# Completely Positive Approximation Property (CPAP)

## Definition

A unital C*-algebra $A$ is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps $\varphi_j$ and $\psi_j$ for $j \in \mathbb{N}$

$$
\begin{tikzcd}
A \arrow[dr, "\varphi_j"'] & & A \\
& M_{n(k)}(\mathbb{C}) \arrow[ur, "\psi_j"'] &
\end{tikzcd}
$$

such that $\psi_j \circ \varphi_j$ converges to $\mathrm{id}_A$ pointwise.

## Lemma

1. *Each $M_n(\mathbb{C})$ is nuclear.*
2. *Direct limits of nuclear algebras are nuclear.*
3. *UHF $\Rightarrow$ AF $\Rightarrow$ nuclear.*
4. *abelian $\Rightarrow$ nuclear.*
5. *$A$ nuclear, $X$ cpct Hausdroff $\Rightarrow$ $C(X, A)$ nuclear.*

Please bear with me - I'll put nuclear algebras on hold for a couple of slides.

# Ultrapowers

According to David Sherman, functional analysts discovered ultrapowers before us.

F. B. Wright, A reduction for algebras of finite type, Ann. of Math. (2) 60 (1954), 560–570.

K. Łos, Quelques remarques, théorèmes et problèmes sur les classes d'efinissables d'algèbres, Mathematical interpretation of formal systems, pp. 98–113. North-Holland Publishing Co., Amsterdam, (1955).

# Ultrapowers II

If $A$ is a C*-algebra and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then let

$$L^\infty(A) = \{(a_n) \in A^{\mathbb{N}} | \sup_n \|a_n\| < \infty\}$$

and

$$c_0(\mathcal{U}) = \{(a_n) \in L^\infty(A) : \lim_{n \to \mathcal{U}} \|a_n\| = 0\}.$$

# Ultrapowers II

If $A$ is a C*-algebra and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then let

$$L^\infty(A) = \{(a_n) \in A^{\mathbb{N}} | \sup_n \|a_n\| < \infty\}$$

and

$$c_0(\mathcal{U}) = \{(a_n) \in L^\infty(A) : \lim_{n\to\mathcal{U}} \|a_n\| = 0\}.$$

The *ultrapower* of $A$ is

$$\prod_{\mathcal{U}} A = L^\infty(A)/c_0(\mathcal{U}),$$

usually denoted $A^{\mathcal{U}}$ by operator algebraists.

# Here is a sample of what I originally planned to talk about

In the following identify $B$ with its diagonal copy in $\prod_{\mathcal{U}} B$.

# Here is a sample of what I originally planned to talk about

In the following identify $B$ with its diagonal copy in $\prod_{\mathcal{U}} B$.

**Exercise**

*Assume $A$ is a subalgebra of a separable algebra $B$, $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, and the ultrapower $\prod_{\mathcal{U}} B$ has automorphisms $\Phi_n$ for $n \in \mathbb{N}$ such that (identifying $A$ and $B$ with their diagonal copies in the ultrapower)*

1. *$\Phi_n$ fixes all elements of $A$,*
2. $\lim_{n\to\infty} \operatorname{dist}(\Phi_n(b), \prod_{\mathcal{U}} A) = 0$ *for all $b \in B$.*

*Then $A \cong B$.*

# Here is a sample of what I originally planned to talk about

In the following identify $B$ with its diagonal copy in $\prod_{\mathcal{U}} B$.

**Exercise**

*Assume $A$ is a subalgebra of a separable algebra $B$, $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, and the ultrapower $\prod_{\mathcal{U}} B$ has automorphisms $\Phi_n$ for $n \in \mathbb{N}$ such that (identifying $A$ and $B$ with their diagonal copies in the ultrapower)*

1. *$\Phi_n$ fixes all elements of $A$,*
2. $\lim_{n\to\infty} \operatorname{dist}(\Phi_n(b), \prod_{\mathcal{U}} A) = 0$ *for all $b \in B$.*

*Then $A \cong B$.*

This is used e.g., to characterize C*-algebras $A$ such that $A \otimes \mathcal{Z} \cong A$ ($\mathcal{Z}$ is the notorious Jiang–Su algebra).

# Some unnerving facts

**Theorem (Junge–Pisier, 1995)**

*There is a finite set $F \subseteq \mathcal{B}(H)$ such that any C*-algebra $A$ such that $F \subseteq A \subseteq \mathcal{B}(H)$ is not nuclear.*

# Some unnerving facts

**Theorem (Junge–Pisier, 1995)**

*There is a finite set $F \subseteq \mathcal{B}(H)$ such that any C*-algebra $A$ such that $F \subseteq A \subseteq \mathcal{B}(H)$ is not nuclear.*

Nuclear algebras form a 'nonstationary set!'

# Some unnerving facts

**Theorem (Junge–Pisier, 1995)**

*There is a finite set $F \subseteq \mathcal{B}(H)$ such that any C*-algebra $A$ such that $F \subseteq A \subseteq \mathcal{B}(H)$ is not nuclear.*

Nuclear algebras form a 'nonstationary set!'

**Lemma**

*An ultrapower of a UHF algebra is not nuclear.*

# Some unnerving facts

### Theorem (Junge–Pisier, 1995)

*There is a finite set $F \subseteq \mathcal{B}(H)$ such that any C*-algebra $A$ such that $F \subseteq A \subseteq \mathcal{B}(H)$ is not nuclear.*

Nuclear algebras form a 'nonstationary set!'

### Lemma

*An ultrapower of a UHF algebra is not nuclear.*

Nuclear algebras are not axiomatizable!
(And the same applies to UHF, AF, AI, AT, AH,. . . ).

# UHF algebras revisited

**Lemma**

*A separable C*-algebra is UHF if and only if it is LM (locally matricial), i.e., if*
*Every finite $F \subseteq A$ is $\varepsilon$-included in some full matrix subalgebra of $A$, for every $\varepsilon > 0$.*

# UHF algebras revisited

## Lemma

*A separable C*-algebra is UHF if and only if it is LM (locally matricial), i.e., if*
*Every finite $F \subseteq A$ is $\varepsilon$-included in some full matrix subalgebra of $A$, for every $\varepsilon > 0$.*

## Proposition

*For every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a type $\mathbf{t}_\varepsilon(x_0, \ldots, x_{n-1})$ in the theory of C*-algebras over $\emptyset$ such that in every C*-algebra $A$, type $\mathbf{t}_\varepsilon$ is realized by $a_0, a_1, \ldots, a_{n-1}$ iff no full matrix subalgebra $\varepsilon$-includes $\{a_0, \ldots, a_{n-1}\}$.*

# Glimm revisited

## Corollary

*There is a sequence of types* $\mathbf{t}_{1/k}$ *for* $k \in \mathbb{N}$ *such that a C*-algebra* $A$ *is UHF iff it omits all of those types.*

# Glimm revisited

### Corollary

*There is a sequence of types* $\mathbf{t}_{1/k}$ *for* $k \in \mathbb{N}$ *such that a C*-algebra* $A$ *is UHF iff it omits all of those types.*

Proofs of both the above and the following use a bit of what I called 'stability' in my first talk.

# Glimm revisited

### Corollary

*There is a sequence of types $\mathbf{t}_{1/k}$ for $k \in \mathbb{N}$ such that a C*-algebra $A$ is UHF iff it omits all of those types.*

Proofs of both the above and the following use a bit of what I called ‘stability’ in my first talk.

### Theorem (Glimm, 1960)

*Separable unital C*-algebras that omit all $\mathbf{t}_{1/k}$ are isomorphic iff they are elementarily equivalent.*

# Glimm revisited

### Corollary

*There is a sequence of types $\mathbf{t}_{1/k}$ for $k \in \mathbb{N}$ such that a C*-algebra A is UHF iff it omits all of those types.*

Proofs of both the above and the following use a bit of what I called 'stability' in my first talk.

### Theorem (Glimm, 1960)

*Separable unital C*-algebras that omit all $\mathbf{t}_{1/k}$ are isomorphic iff they are elementarily equivalent.*

(Not surprisingly, this fails in the nonseparable case by F.–Katsura.)

# Revisiting AF

## Proposition

*There is a sequence of types* $\mathbf{s}_{1/k}$ *for* $k \in \mathbb{N}$ *such that a C*-algebra A is AF iff it omits all of those types.*

# Revisiting AF

## Proposition

*There is a sequence of types $\mathbf{s}_{1/k}$ for $k \in \mathbb{N}$ such that a C*-algebra A is AF iff it omits all of those types.*

# Revisiting AF...but not Elliott

## Proposition

*There is a sequence of types* $\mathbf{s}_{1/k}$ *for* $k \in \mathbb{N}$ *such that a C*-algebra* $A$ *is AF iff it omits all of those types.*

## Proposition

*There are separable, unital AF algebras that are elementariy equivalent but nonisomorphic.*

# Revisiting AF... but not Elliott

## Proposition

*There is a sequence of types* $\mathbf{s}_{1/k}$ *for* $k \in \mathbb{N}$ *such that a C*-algebra* $A$ *is AF iff it omits all of those types.*

## Proposition

*There are separable, unital AF algebras that are elementariy equivalent but nonisomorphic.*

## Proof.

Let $\mathcal{S}$ be the set of all sentences in the language of C*-algebras.

$$\begin{tikzcd}
\begin{array}{l}\text{separable}\\ \text{unital}\\ \text{AF algebras}\end{array} \arrow[rr, "A \mapsto \mathrm{Th}(A)"] \arrow[dr, leftrightarrow] & & F(\mathcal{S}) \\
& \begin{array}{l}K_0 \text{ groups}\\ \text{of AF algebras}\\ \text{(dimension groups)}\end{array} \arrow[ur, "/" marking] &
\end{tikzcd}$$

# K-theory is good

**Problem**

*Is there a model-theoretic interpretation of Elliott's theorem?*

# K-theory is good

**Problem**

*Is there a model-theoretic interpretation of Elliott's theorem?*

All known obstructions to $\aleph_1$-saturation of the Calkin algebra and other corona algebras are of K-theoretic nature. (F.–B. Hart–N. C. Phillps).

# K-theory is good

### Problem

*Is there a model-theoretic interpretation of Elliott's theorem?*

All known obstructions to $\aleph_1$-saturation of the Calkin algebra and other corona algebras are of K-theoretic nature. (F.–B. Hart–N. C. Phillps).

### Question

*Is K-theory the only obstruction to $\aleph_1$-saturation of the Calkin algebra?*

# What about the nuclearity?

**Definition**

A unital C*-algebra $A$ is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps $\varphi_j$ and $\psi_j$ for $j \in \mathbb{N}$

$$\begin{tikzcd} A \arrow[dr, "\varphi_j"'] & & A \\ & M_{n(k)}(\mathbb{C}) \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

such that $\varphi_j \circ \psi_j$ converges to $\mathrm{id}_A$ pointwise.

# What about the nuclearity?

## Definition

A unital C*-algebra $A$ is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps $\varphi_j$ and $\psi_j$ for $j \in \mathbb{N}$

$$A \xrightarrow{\varphi_j} M_{n(k)}(\mathbb{C}) \xrightarrow{\psi_j} A$$

such that $\varphi_j \circ \psi_j$ converges to $\mathrm{id}_A$ pointwise.

## Conjecture

*There is a sequence of types such that the nuclear algebras are exactly the C*-algebras omitting those types.*

# What about the nuclearity?

## Definition

A unital C*-algebra $A$ is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps $\varphi_j$ and $\psi_j$ for $j \in \mathbb{N}$

$$\begin{tikzcd} A \arrow[dr, "\varphi_j"'] & & A \\ & M_{n(k)}(\mathbb{C}) \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

such that $\varphi_j \circ \psi_j$ converges to $\mathrm{id}_A$ pointwise.

## Conjecture

*There is a sequence of types such that the nuclear algebras are exactly the C*-algebras omitting those types.*

## Thesis

*We have only scratched the surface.*


C STARR
GMC