Long-low iterations / matrix forcing

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[2]this paper initiated at Fields Oct 2012
see forthcoming F1222

Forcing at Fields

Goal

we want to force a model of $\mathfrak{t} < \mathfrak{h} = \kappa < \mathfrak{s} = \lambda$
and see where we can put $\mathfrak{b}$

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Definition

We can define $\mathfrak{h}$ as the minimum cardinal for which there is a sequence $\langle \mathcal{I}_\xi : \xi \in \mathfrak{h} \rangle$ of $\subset^*$-dense ideals on $\mathcal{P}(\omega)$ with empty intersection (or maybe intersection equal to $[\omega]^{<\aleph_0}$)

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Shelah: the forcing $\mathcal{Q}_{Bould}$ with countable support to first get
$\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$

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family of special ccc subposets of $\mathcal{Q}_{Bould}$: we'll call $\mathbb{Q}_{207}$
first used by Fischer-Steprans

Brief history

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Shelah [1983] in Boulder proceedings introduced $\mathcal{Q}_{Bould}$ to obtain $\omega_1 = \mathfrak{b} < \mathfrak{s} = \mathfrak{a}$.

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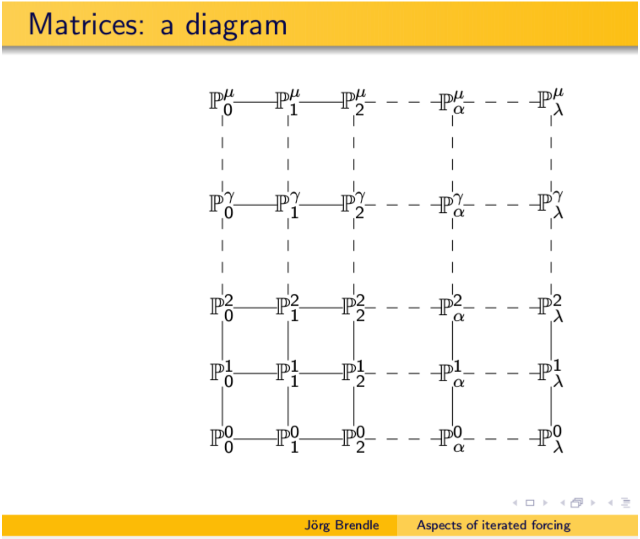
Notes

It was shown in Brendle-Raghavan [2014] that $\mathcal{Q}_{Bould}$ can be factored as countably closed * ccc Mathias
(similar to Fischer-Steprans but still limits to κ^+).
Brendle delivered a beautiful workshop on matrix forcing at Czech WS 2010.

a matrix iteration $\langle \mathbb{P}(\alpha,\gamma), \mathbb{Q}(\alpha,\gamma) : \gamma \leq \mu \,, \alpha < \lambda \rangle$

in case you don't know what a matrix looks like



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This means $\dot{Y}$ won't know about even $\mathbb{P}(0, i + 1)$ and so gives us a chance to keep a cardinal invariant small

illustrative examples

Let us look at two examples where $\mathbb{P}(0, i)$ is $\mathrm{FS}_{j \le i} \mathcal{H}_j$ adding $\langle H_i^0 : i < \kappa \rangle$

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iterate Hechler up every column

If, for all $\alpha > 0$ and i, $\dot{\mathbb{Q}}(\alpha,i)$ is $\left(\bigcup_{j<i}\dot{\mathbb{Q}}(\alpha,j)\right) * \mathcal{H}$

up each column, iteratively add Hechler reals

then we get a model of $\mathfrak{b} = \kappa < \mathfrak{d} = \lambda$ (and $\mathfrak{h} = \omega_1$)

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remark

In first case, it is obvious that $\mathbb{P}(\alpha,i)<_c\mathbb{P}(\alpha,i+1)$, but not so much in the second case (more on this later)

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In fact, let us notice that $\mathcal{H}^{V_{\alpha,i}} \not\lessdot_c \mathcal{H}^{V_{\alpha,i+1}}$,

but it IS

the construction of the chain $\{\mathbb{Q}_{\alpha,i} : i < \kappa\}$ that controls things.

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If γ is a limit and we have an increasing sequence $\{\underline{\mathbf{P}}^\delta : \delta < \gamma\}$ of matrices, then the union $\underline{\mathbf{P}}^\gamma$ extends canonically to a γ-matrix

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The union, $\bigcup_{\delta<\gamma} \underline{\mathbf{P}}^\delta$ will be a list $\{\mathbb{P}(\alpha, i) : i \leq \kappa, \alpha < \gamma\}$. For each $i < \kappa$, $\mathbb{P}(\gamma, i)$ must equal $\bigcup_{\delta<\gamma} \mathbb{P}(\delta, i)$.
And, as needed, we have $\mathbb{P}(\gamma, j) <_c \mathbb{P}_{\gamma,i}$ $\quad (j < i \leq \kappa)$

Lemma (Brendle-Fischer)

Suppose $\mathbb{P} <_c \mathbb{P}'$, *and* $\mathbb{Q}$ *is a* $\mathbb{P}$*-name and* $\mathbb{Q}'$ *is a* $\mathbb{P}'$*-name.*

For $\mathbb{P} * \mathbb{Q} <_c \mathbb{P}' * \mathbb{Q}'$, *we need*

every $\mathbb{P}$*-name of a maximal antichain of* $\mathbb{Q}$ *is also forced by* $\mathbb{P}'$ *to be a maximal antichain of* $\mathbb{Q}'$.

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Corollary (for successor $\alpha < \lambda$)

If $\underline{\mathbf{P}}^\alpha$ *is given, and if* $\mathcal{Y}_\alpha$ *is a* $\mathbb{P}_{\alpha,i_\alpha}$*-name of a sfip family, we can let* $\mathbb{Q}_{\alpha,j}$ *be trivial for* $j < i_\alpha$ *and let* $\mathbb{Q}_{\alpha,i} = Q(\mathcal{Y}_\alpha)$ *for* $j \geq i_\alpha$ *with generic set* $\dot{A}_\alpha$*.*

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$\mathcal{I}_i = ideal\langle\{\dot{A}_\alpha : i_\alpha = i\}\rangle$ *towards* $\mathfrak{h} \leq \kappa$.
With more tedious bookkeeping, $\mathcal{I}_j \supset \mathcal{I}_i$ *(for* $j < i$*)*

Proposition (Ihoda-Shelah, 1988)

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If $\underline{\mathbf{P}}^{\alpha}$ is given, then we can let $\underline{\mathbf{P}}^{\alpha+1}$ be constructed with $\dot{\mathbb{Q}}_{\alpha,i} = \mathcal{H}$ for all $i \leq \kappa$.

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Definition (fundamental Ind. Hyp.)

By induction on $\gamma < \lambda$, when building $\underline{\mathbf{P}}^{\gamma}$ and setting

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Corollary (Baumgartner-Dordal)

When $cf(\alpha) = \kappa$ and we let $\dot{\mathbb{Q}}_{\alpha,i} = \mathcal{H}$, we preserve Ind Hyp.

Now we discuss $\mathcal{Q}_{Bould}$ and $\mathbb{Q}_{207}$

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finite working part

Elements $q = (w^q, T^q)$ of $\mathcal{Q}_{Bould}$, like all our posets,
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stronger Induction Hypothesis seems necessary

Before, or even if, discussing what such a $(w, T) \in \mathcal{Q}_{Bould}$ looks like, I seemed to need a stronger hypothesis on $\underline{\mathbf{P}}^{\mu}$ in order to be able to construct $\dot{Q}_{\mu,i} \in \mathbb{Q}_{207}$ to do the job.

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new Ind. Hyp. : Γ_i^μ-pure

For any dense set $D \subset P_{\mu,i+1}$ and any Γ_i^μ-fan $\langle p_0, p_1, \dots, p_n \rangle$, there is an extension Γ_i^μ-fan $\langle p_0, \bar{p}_1, \dots, \bar{p}_n \rangle$ such that $\{\bar{p}_1, \dots, \bar{p}_n\} \subset D$.

this is a good Ind. Hyp.

Lemma (assume Γ_i^μ-pure)

By induction on μ, if $\dot{Y}$ is a $\mathbb{P}_{\mu,i}$-name and $\langle p_0, p_1, \cdots, p_n\rangle$ is a Γ_i^μ-fan, then, for $1 \leq j, k \leq n$, integer y,

$$p_j \Vdash y \in \dot{Y} \quad \text{iff} \quad p_k \Vdash y \in \dot{Y}$$

and $$p_j \perp p \quad \text{iff} \quad p_k \perp p \text{ for each } p \in \mathbb{P}_{\mu,i}$$

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Corollary

If $p_0 \in \mathbb{P}_{\mu,i}$ and $\dot{Y}$ is a $\mathbb{P}_{\mu,i}$-name, and $p_0 \Vdash \dot{Y} \subset \dot{A}_\alpha \cup m$ for some $\alpha \in \Gamma_i^\mu$, then $p_0 \Vdash \dot{Y}$ is finite. thus preserves Ind. Hyp.

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Hechler preserves the Ind. Hyp Γ_i^μ-pure

Lemma (Baumgartner-Dordal)

If $D \subset \mathcal{H}$ is dense, there is a function $rk_D : \omega^{<\omega\uparrow} \mapsto \omega_1$ such that $rk(s) = 0$ if there is a g with $(s, g) \in D$, and $rk(s) = \alpha > 0$ if there is an ℓ such that for each n, there is an $(s_n, g + n) < (s, g + n)$ with $s_n \in \omega^{\ell\uparrow}$ and $rk(s_n) < \alpha$.

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Now let $\dot{D}$ be a $\mathbb{P}_{\mu,i+1}$-name of a dense subset of $\mathcal{H}$. Also, let $\langle p_0, p_1, \ldots, p_n \rangle$ be any Γ_i^μ-fan.

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For $\Gamma_i^{\mu+1}$, we have to find an extension fan $\langle p_0, \bar{p}_1, \ldots, \bar{p}_n \rangle$ so that $\bar{p}_k \restriction \mu \Vdash p_k(\mu) \in \dot{D}$ for all $1 \leq k \leq n$.

proof continued

We may assume that $p_0(\mu) = (s_0, \dot{g}_0)$, which means that, we can simply assume that $p_j(\mu) = (s_0, \dot{g}_0)$ for all $j \leq n$

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There is an extension fan $\langle p_0, \bar{p}_1, \cdots, \bar{p}_n \rangle$ so that each $\bar{p}_k$ forces a value on $\dot{g}_0 \restriction \ell_0$ and $\bar{p}_1$ picks an s_1 so that each $\bar{p}_k \Vdash (s_1, \dot{g}_0) < (s_0, \dot{g}_0)$ and $\bar{p}_1$ forces that $rk(s_1) = \alpha_1 < \alpha_0$.

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Repeat this finitely many times (as rank descends) we end up with there being a $\dot{g}_1$ such that $\bar{p}_1 \Vdash (s_1, \dot{g}_1) \in \dot{D}$ and, for all $1 \leq k \leq n$ and $\bar{p}_k \Vdash (s_1, \dot{g}_1) < (s_0, \dot{g}_0)$.

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Make the same steps (keep extending the fan) so that we then have an s_2 and $\dot{g}_2$ so that $\bar{p}_2 \Vdash (s_2, \dot{g}_2) \in \dot{D}$, and each $\bar{p}_k \Vdash (s_2, \dot{g}_2) < (s_1, \dot{g}_1)$.

okay, finally back to $\mathcal{Q}_{Bould}$

Definition (from Avraham)

h is a log-measure on a set e if $h(k) = 0$ for all $k \in e$ and if $h(e_1 \cup e_2) > \ell > 0$, then one of $h(e_1), h(e_2)$ is at least ℓ.

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Definition

the log-measure (e, h) is <u>built from</u> the sequence $\langle (e_1, h_1), \ldots, (e_n, h_n) \rangle$ ($\max(e_k) < \min(e_{k+1})$) if $e \subset (e_1 \cup \cdots e_n)$ and if $x \subset e$ is h-positive, then there is a k such that $x \cap e_k$ is h_k-positive

Definition

$q = (w^q, T^q) \in Q_{Bould}$ if

$T^q = \langle t_k = (e_k, h_k) : k \in \omega \rangle$ and

$\max(e_k) < \min(e_{k+1})$ and $\liminf\{h_k(e_k) : k \in \omega\} = \infty$

We let $int(T) = \bigcup_k int(t_k) = \bigcup_k e_k$ and

$(w_2, T_2) < (w_1, T_1)$ if each t^2_k is built from members of T_1

and there is an ℓ such that

$w_1 = w_2 \cap \min(int(t^1_\ell))$ and $w_2 \setminus w_1 \subset int(T_1) \setminus \min(int(t^1_\ell))$

$\mathbb{Q}_{207}$ and $\aleph_1$-directed

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Definition (how to handle $<_c$ for $\mathcal{Q}_{Bould}$)

A subset $Q \subset \mathcal{Q}_{Bould}$ is in $\mathbb{Q}_{207}$ if it is closed under finite changes, the subfamily $\{q \in Q : w^q = \emptyset\}$ is directed, and

whenever $\{(w_n, T_n) : n \in \omega\}$ is pre-dense, there is a single T such that, $(\emptyset, T) \in Q$ and for each n, there is an ℓ_n such that $(w_n, T \setminus \ell_n) < (w_n, T_n)$. (we made it upward absolute)

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Lemma (Fischer-Steprans partially)

If $Q \in \mathbb{Q}_{207}$ and P is ccc, and $\Vdash_P Q \subset Q_1 \in \mathbb{Q}_{207}$ then $Q <_c P * Q_1$. Furthermore, if $Q \subset \mathcal{Q}_{Bould}$ is closed under finite changes and weakly centered, and P is ccc, then there is a $P * \mathcal{C}_{2^\omega}$-name $\dot{Q}_1$ such that $\Vdash Q \subset \dot{Q}_1 \in \mathbb{Q}_{207}$ and adds an unsplit real over V.

Finishing the construction of $\underline{\mathbf{P}}^{\lambda}$

Lemma

Let $\mu < \lambda$ be a limit of cofinality $\neq\kappa$ and assume that $\mathbb{P}_{\mu,i+1}$ is a Γ_i^{μ}-pure extension of $\mathbb{P}_{\mu,i}$. Assume further that $\dot{Q}_{\mu,i}$ is a $\mathbb{P}_{\mu,i} * \mathcal{C}_{2^\omega}$-name of a member of $\mathbb{Q}_{207}$. Then there is a $\mathbb{P}_{\mu,i+1} * \mathcal{C}_{2^\omega+2^\omega}$-name $\dot{Q}_{\mu,i+1}$ that is forced to be a member of $\mathbb{Q}_{207}$ and such that $\mathbb{P}_{\mu+1,i+1}$ is a $\Gamma_i^{\mu+1}$-pure extension of $\mathbb{P}_{\mu+1,i}$. In addition, $\dot{Q}_{\mu,i+1}$ can be chosen so that it adds an unsplit real over the extension by $\mathbb{P}_{\mu,i}$.

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Remark

When handling a pre-dense $\{(u_n, T_n) : n \in \omega\}$ (in $V[G_{\mu,i}]$) from $\dot{Q}_{\mu,i}$, towards extending into $\mathbb{Q}_{207}$ we may not be able to do so (Cohen forcing) while keeping things $\Gamma_{\mu,i}$-pure

but then we Cohen force with fans as side-conditions to add to $\dot{Q}_{\mu,i+1}$ in a Γ_i^μ-pure way and destroy the pre-density.

conclusion and questions

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There is an easy trick to lower $\mathfrak{t}$ to ω_1 (or any other value) while leaving others the same.

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Question

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