A disjoint union theorem for trees

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Fields Institute, 2015

Finite disjoint union Theorem

Theorem (Folkman)

For every pair of positive integers m and r there is integer n_0 such that for every r-coloring of the power-set $\mathcal{P}(X)$ of some set X of cardinality at least n_0, there is a family $\mathbf{D} = (D_i)_{i=1}^m$ of pairwise disjoint nonempty subsets of X such that the family

$$\mathcal{U}(\mathbf{D}) = \left\{ \bigcup_{i \in I} D_i : \emptyset \neq I \subseteq \{1, 2, ..., m\} \right\}$$

of non-empty unions is monochromatic.

Infinite disjoint union Theorem

Theorem (Carlson-Simpson)

For every finite Souslin measurable coloring of the power-set $\mathcal{P}(\omega)$ of ω, there is a sequence $\mathbf{D} = (D_n)_{n<\omega}$ of pairwise disjoint subsets of the natural numbers such that the set

$$\mathcal{U}(\mathbf{D}) = \Big\{ \bigcup_{n\in M} D_n : \ M \textit{ is a non-empty subset of } \omega \Big\}$$

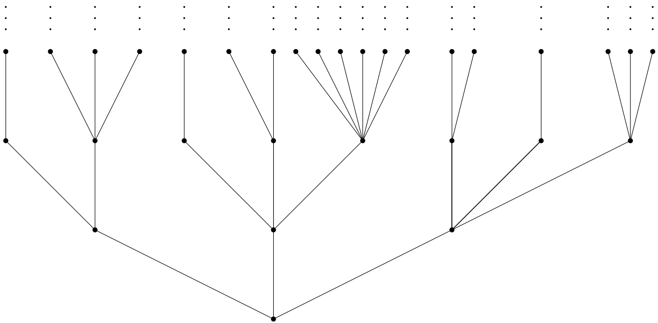
is monochromatic.

Trees

A **tree** is a partially ordered set $(T, \leq_T)$ such that

$$\mathrm{Pred}_T(t) = \{s \in T : s <_T t\}$$

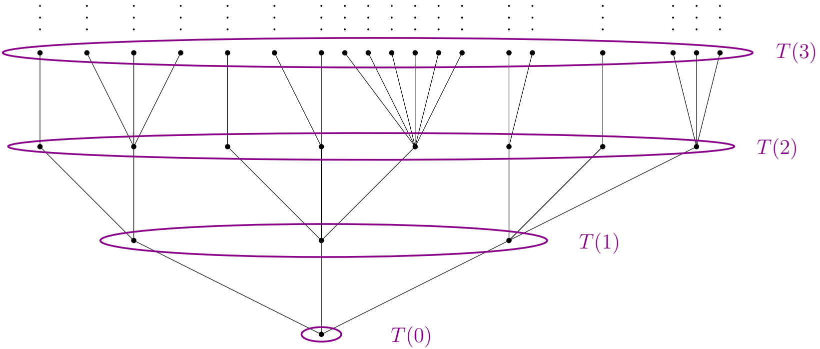
is is finite and totally ordered for all t in T.
We consider only **uniquely rooted and finitely branching trees with no maximal nodes**.



Levels

For $n < \omega$, the **n-th level of T**, is the set

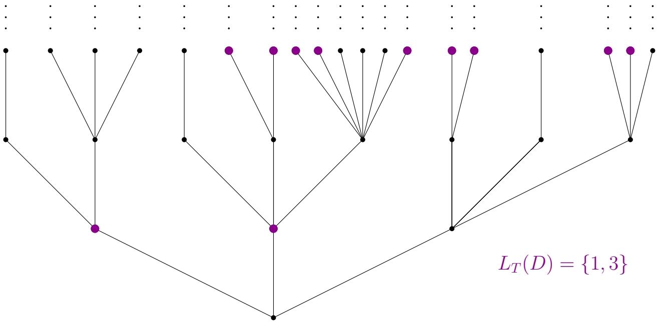
$$T(n) = \{t \in T : |\mathrm{Pred}_T(t)| = n\}.$$



Level set

For a subset D of T, we define its **level set**

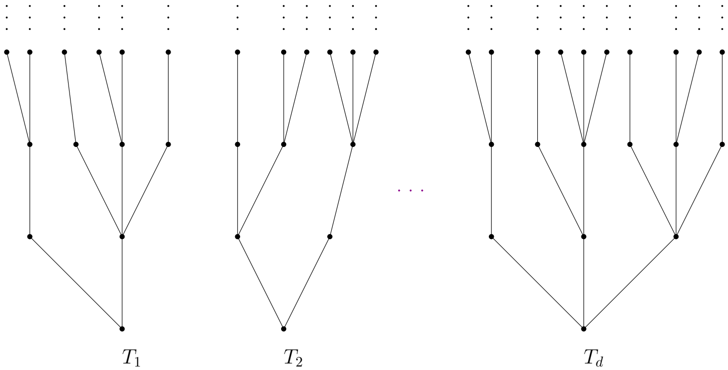
$$L_T(D) = \{n \in \omega : D \cap T(n) \neq \emptyset\}$$



From now on, fix an integer $d \geq 1$.
A **vector tree**

$$\mathbf{T} = (T_1, ..., T_d)$$

is a d-sequence of uniquely rooted and finitely branching trees with no maximal nodes.



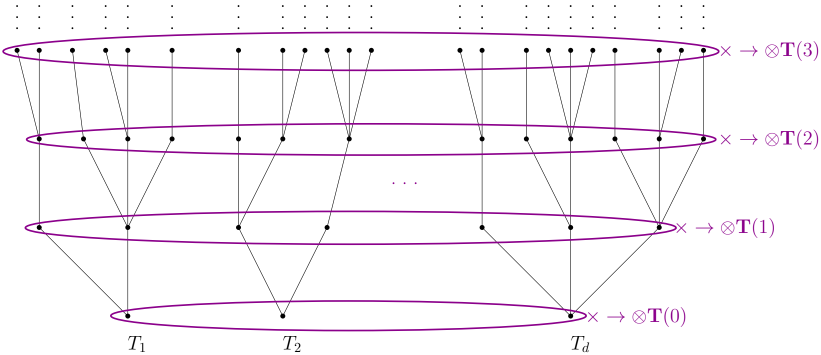
Level products

For a vector tree $\mathbf{T} = (T_1, \ldots, T_d)$ we define its **level product** as

$$\otimes\mathbf{T} = \bigcup_{n<\omega} T_1(n) \times \ldots \times T_d(n)$$

The n**-th level of the level product** of $\mathbf{T}$ is

$$\otimes\mathbf{T}(n) = T_1(n) \times \ldots \times T_d(n).$$



Vector trees

Let $\mathbf{T} = (T_1, ..., T_d)$ a vector tree.
For $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{s} = (s_1, ..., s_d)$ in $\otimes\mathbf{T}$, set

$$\mathbf{t} \leq_{\mathbf{T}} \mathbf{s} \text{ iff } t_i \leq_{T_i} s_i \text{ for all } i = 1, ..., d.$$

For $\mathbf{t} = (t_1, ..., t_d)$ in $\otimes\mathbf{T}$, we define

$$\mathrm{Succ}_{\mathbf{T}}(\mathbf{t}) = \{\mathbf{s} \in \otimes\mathbf{T} : \mathbf{t} \leq_{\mathbf{T}} \mathbf{s}\}$$

Vector subsets and dense vector subsets products

A sequence $\mathbf{D} = (D_1, ..., D_d)$ is called a **vector subset** of $\mathbf{T}$ if D_i is a subset of T_i for all $i = 1, ..., d$ and

$$L_{T_1}(D_1) = ... = L_{T_d}(D_d).$$

For a vector subset $\mathbf{D}$ of $\mathbf{T}$ we define its **level product**

$$\otimes\mathbf{D} = \bigcup_{n<\omega} (T_1(n) \cap D_1) \times ... \times (T_d(n) \cap D_d).$$

For $\mathbf{t} \in \otimes\mathbf{T}$, a vector subset $\mathbf{D}$ of $\mathbf{T}$ is **t-dense,** ,

$$(\forall n)(\exists m)(\forall \mathbf{s} \in \otimes\mathbf{T}(n) \cap \mathrm{Succ}_{\mathbf{T}}(\mathbf{t})(\exists \mathbf{s}' \in \otimes\mathbf{T}(m) \cap \otimes\mathbf{D}) \ \ \mathbf{s} \leq_{\mathbf{T}} \mathbf{s}'.$$

$\mathbf{D}$ is called **dense** if it is root$(\otimes\mathbf{T})$-dense.

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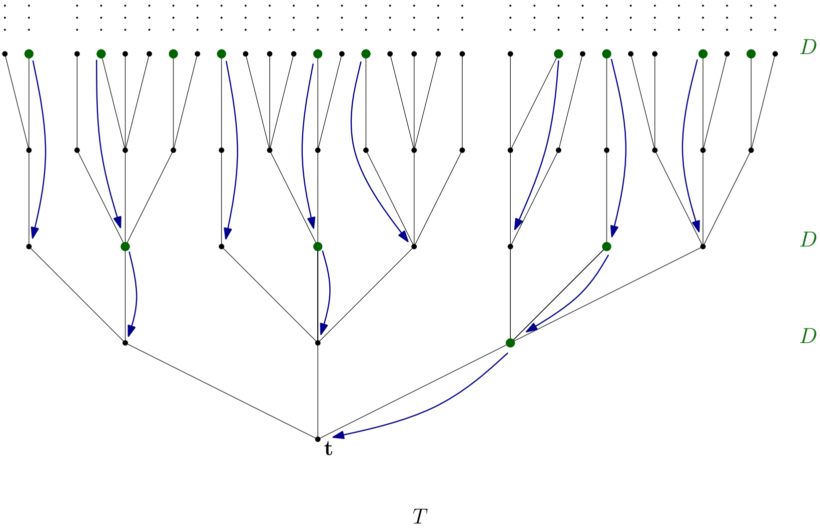
Dense vector subset



$$(\forall n)(\exists m)(\forall \mathbf{s} \in \otimes \mathbf{T}(n) \cap \mathrm{Succ}_{\mathbf{T}}(\mathbf{t})(\exists \mathbf{s}' \in \otimes \mathbf{T}(m) \cap \otimes \mathbf{D})\ \ \mathbf{s} \leq_{\mathbf{T}} \mathbf{s}'.$$

Theorem (Halpern–Läuchli)

Let $\mathbf{T}$ be a vector tree. Then for every dense vector subset $\mathbf{D}$ of $\mathbf{T}$ and every subset $\mathcal{P}$ of $\otimes\mathbf{D}$, there exists a vector subset $\mathbf{D}'$ of $\mathbf{D}$ such that either

(i) *$\otimes\mathbf{D}'$ is a subset of $\mathcal{P}$ and $\mathbf{D}'$ is a dense vector subset of $\mathbf{T}$, or*

(ii) *$\otimes\mathbf{D}'$ is a subset of $\mathcal{P}^c$ and $\mathbf{D}'$ is a $\mathbf{t}$-dense vector subset of $\mathbf{T}$ for some $\mathbf{t}$ in $\otimes\mathbf{T}$.*

Let $\mathbf{T}$ be a vector tree. We define

$$\mathcal{U}(\mathbf{T}) = \{U \subseteq \otimes\mathbf{T} : \ U \text{ has a minimum}\}.$$

We let $\mathcal{U}(\mathbf{T})$ take its topology from $\{0,1\}^{\otimes\mathbf{T}}$.
Let $\mathbf{D}$ be a vector subset of $\mathbf{T}$.
A **D-subspace** of $\mathcal{U}(\mathbf{T})$ is a family

$$\mathbf{U} = (U_{\mathbf{t}})_{\mathbf{t}\in\otimes\mathbf{D}}$$

such that

1. $U_{\mathbf{t}} \in \mathcal{U}(\mathbf{T})$ for all $\mathbf{t} \in \otimes\mathbf{D}$,
2. $U_{\mathbf{s}} \cap U_{\mathbf{t}} = \emptyset$ for $\mathbf{s} \neq \mathbf{t}$,
3. $\min U_{\mathbf{t}} = \mathbf{t}$ for all $\mathbf{t} \in \otimes\mathbf{D}$.

The span of a subspace

For a subspace $\mathbf{U} = (U_{\mathbf{t}})_{\mathbf{t}\in\otimes\mathbf{D}(\mathbf{U})}$ we define its **span** by

$$[\mathbf{U}] = \Big\{ \bigcup_{\mathbf{t}\in\Gamma} U_{\mathbf{t}} : \ \Gamma \subseteq \otimes\mathbf{D}(\mathbf{U}) \Big\} \cap \mathcal{U}(\mathbf{T})$$
$$= \Big\{ \bigcup_{\mathbf{t}\in\Gamma} U_{\mathbf{t}} : \ \Gamma \subseteq \otimes\mathbf{D}(\mathbf{U}) \text{ and } \Gamma \in \mathcal{U}(\mathbf{T}) \Big\}.$$

If $\mathbf{U}$ and $\mathbf{U}'$ are two subspaces of $\mathcal{U}(\mathbf{T})$, we say that $\mathbf{U}'$ **is a subspace of** $\mathbf{U}$, and write $\mathbf{U}' \leq \mathbf{U}$, if $[\mathbf{U}'] \subseteq [\mathbf{U}]$.

Remark

$\mathbf{U}' \leq \mathbf{U}$ implies that $\mathbf{D}(\mathbf{U}')$ is a vector subset of $\mathbf{D}(\mathbf{U})$.

Disjoint union Theorem for vector trees

Theorem

Let $\mathbf{T}$ *be a vector tree and* $\mathcal{P}$ *a Souslin measurable subset of* $\mathcal{U}(\mathbf{T})$. *Also let* $\mathbf{D}$ *be a dense vector subset of* $\mathbf{T}$ *and* $\mathbf{U}$ *a* $\mathbf{D}$*-subspace of* $\mathcal{U}(\mathbf{T})$. *Then there exists a subspace* $\mathbf{U}'$ *of* $\mathcal{U}(\mathbf{T})$ *with* $\mathbf{U}' \leq \mathbf{U}$ *such that either*

(i) $[\mathbf{U}']$ *is a subset of* $\mathcal{P}$ *and* $\mathbf{D}(\mathbf{U}')$ *is a dense vector subset of* $\mathbf{T}$, *or*

(ii) $[\mathbf{U}']$ *is a subset of* $\mathcal{P}^c$ *and* $\mathbf{D}(\mathbf{U}')$ *is a* $\mathbf{t}$*-dense vector subset of* $\mathbf{T}$ *for some* $\mathbf{t}$ *in* $\otimes\mathbf{T}$.

Corollary (Carlson–Simpson)

For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $\mathbf{D} = (D_n)_{n<\omega}$ of pairwise disjoint subsets of ω such that the set $\mathcal{U}(\mathbf{D})$ is monochromatic.

Let Λ be a finite alphabet. We view the elements of Λ^ω as infinite constant words over Λ. Also let $(v_n)_n$ be a sequence of distinct symbols that do not occur in Λ. An infinite dimensional variable word is a map $f : \omega \to \Lambda \cup \{v_n : n \in \mathbb{N}\}$ such that for every n we have that $f^{-1}(v_n) \neq \emptyset$ and $\max f^{-1}(v_n) < \min f^{-1}(v_{n+1})$. If $(a_n)_n \in \Lambda^\omega$ then by $f((a_n)_n)$ we denote the constant word resulting by substituting each occurrence of v_n by a_n.

Theorem

Let Λ be a finite alphabet. Then for every Souslin measurable coloring of Λ^ω there exists an infinite dimensional word such that the set $\{f((a_n)_n) : (a_n)_n \in \Lambda^\omega\}$ is monochromatic.

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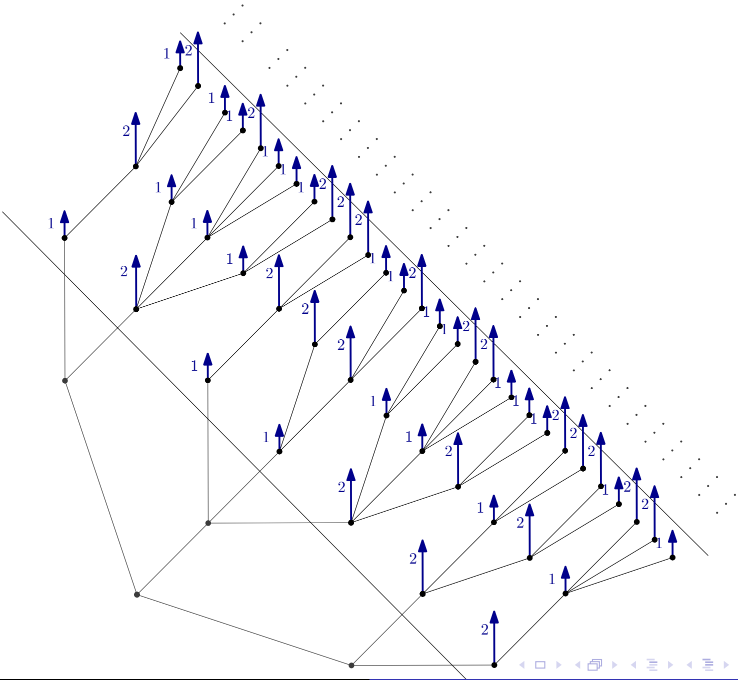
We fix a vector tree $\mathbf{T}$.
Fix a **finite alphabet** Λ.
For $m < n < \omega$, set

$$\mathrm{W}(\Lambda, \mathbf{T}, m, n) = \Lambda^{\otimes \mathbf{T} \upharpoonright [m,n)},$$

where $\otimes \mathbf{T} \upharpoonright [m, n) = \bigcup_{j=m}^{n-1} \otimes \mathbf{T}(j)$. We also set

$$\mathrm{W}(\Lambda, \mathbf{T}) = \bigcup_{m \leq n} \mathrm{W}(\Lambda, \mathbf{T}, m, n).$$



Let $(v_{\mathbf{s}})_{\mathbf{s}\in\otimes\mathbf{T}}$ be a collection of distinct **variables**, set of symbols disjoint from Λ.

Fix a vector level subset $\mathbf{D}$ of $\mathbf{T}$. Let

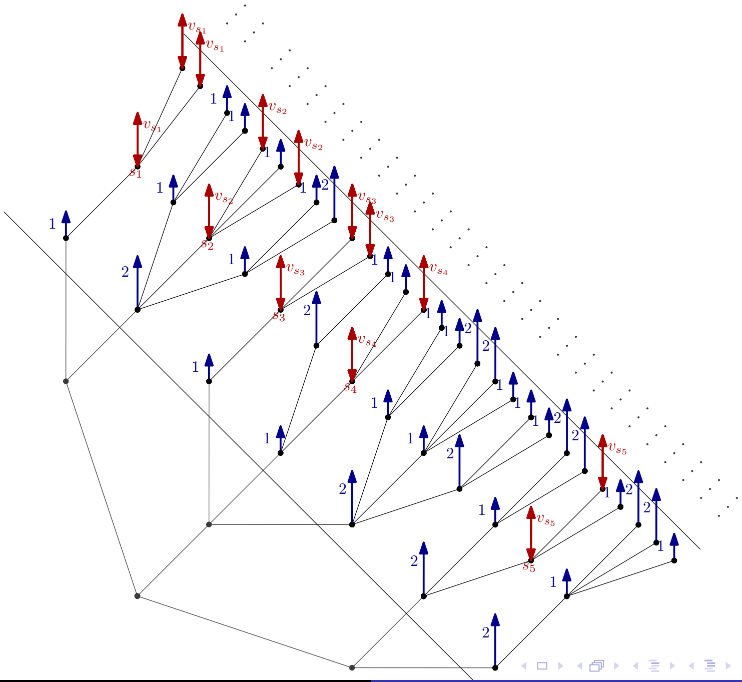
$$\mathrm{W}_v(\Lambda, \mathbf{T}, \mathbf{D}, m, n)$$

to be the set of all functions

$$f : \otimes\mathbf{T} \upharpoonright [m, n) \to \Lambda \cup \{v_{\mathbf{s}} : \mathbf{s} \in \otimes\mathbf{D}\}$$

such that

- The set $f^{-1}(\{v_{\mathbf{s}}\})$ is nonempty and admits $\mathbf{s}$ as a minimum in $\otimes\mathbf{T}$, for all $\mathbf{s} \in \otimes\mathbf{D}$.
- For every $\mathbf{s}$ and $\mathbf{s}'$ in $\otimes\mathbf{D}$, we have $L_{\otimes\mathbf{T}}(f^{-1}(\{v_{\mathbf{s}}\})) = L_{\otimes\mathbf{T}}(f^{-1}(\{v_{\mathbf{s}'}\}))$.



v_{s_1}
s_1
v_{s_2}
s_2
v_{s_3}
s_3
v_{s_4}
s_4
v_{s_5}
s_5

For $f \in \mathrm{W}_v(\Lambda, \mathbf{T}, \mathbf{D}, m, n)$, set

$$\mathrm{ws}(f) = \mathbf{D}, \mathrm{bot}(f) = m \text{ and } \mathrm{top}(f) = n.$$

Moreover, we set

$$\mathrm{W}_v(\Lambda, \mathbf{T}) = \bigcup \Big\{ \mathrm{W}_v(\Lambda, \mathbf{T}, \mathbf{D}, m, n) : m \leq n \text{ and } \mathbf{D} \text{ is a vector level subset of } \mathbf{T} \text{ with } L_{\mathbf{T}}(\mathbf{D}) \subset [m, n) \Big\}.$$

The elements of $\mathrm{W}_v(\Lambda, \mathbf{T})$ are viewed as **variable words over the alphabet Λ.**

For variable words f in $\mathrm{W}_v(\Lambda, \mathbf{T})$ we take **substitutions**:
For every family $\mathbf{a} = (a_{\mathbf{s}})_{\mathbf{s}\in\otimes\mathrm{ws}(f)} \subseteq \Lambda$, let
$f(\mathbf{a}) \in \mathrm{W}(\Lambda, \mathbf{T})$ be the result of substituting for every $\mathbf{s}$ in $\otimes\mathrm{ws}(f)$
each occurrence of $v_{\mathbf{s}}$ by $a_{\mathbf{s}}$, .
Moreover, we set

$$[f]_\Lambda = \{f(\mathbf{a}) : \ \mathbf{a} = (a_{\mathbf{s}})_{\mathbf{s}\in\otimes\mathrm{ws}(f)} \subseteq \Lambda\},$$

the constant span of f.

An infinite sequence $X = (f_n)_{n<\omega}$ in $\mathrm{W}_v(\Lambda, \mathbf{T})$ is a **subspace**, if:

1. $\mathrm{bot}(f_0) = 0$.
2. $\mathrm{bot}(f_{n+1}) = \mathrm{top}(f_n)$ for all $n < \omega$.
3. Setting $D_i = \bigcup_{n<\omega} \mathrm{ws}_i(f_n)$ for all $i = 1, ..., d$, where $\mathrm{ws}(f_n) = (\mathrm{ws}_1(f_n), ..., \mathrm{ws}_d(f_n))$, we have that $(D_1, ..., D_d)$ forms a dense vector subset of $\mathbf{T}$.

For a subspace $X = (f_n)_{n<\omega}$ we define

$$[X]_\Lambda = \Big\{ \bigcup_{q=0}^{n} g_q : \; n < \omega \text{ and } g_q \in [f_q]_\Lambda \text{ for all } q = 0, ..., n \Big\}.$$

For two subspaces X and Y, we write $X \leq Y$ if $[X]_\Lambda \subseteq [Y]_\Lambda$.

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Theorem

Let Λ be a finite alphabet and $\mathbf{T}$ a vector tree. Then for every finite coloring of the set of the constant words $\mathrm{W}(\Lambda, \mathbf{T})$ over Λ and every subspace X of $\mathrm{W}(\Lambda, \mathbf{T})$ there exists a subspace X' of $\mathrm{W}(\Lambda, \mathbf{T})$ with $X' \leq X$ such that the set $[X']_\Lambda$ is monochromatic.

A Ramsey space of sequences of words

Let $\mathrm{W}^\infty(\Lambda, \mathbf{T})$, be the set of all sequences $(g_n)_{n<\omega}$ in $\mathrm{W}(\Lambda, \mathbf{T})$ such that:

1. $\mathrm{bot}(g_0) = 0$ and
2. $\mathrm{bot}(g_{n+1}) = \mathrm{top} g_n$ for all $n < \omega$.

For a subspace X, we set

$$[X]^\infty_\Lambda = \{(g_n)_{n<\omega} \in \mathrm{W}^\infty(\Lambda, \mathbf{T}) : (\forall n < \omega) \bigcup_{q=0}^{n} g_q \in [X]_\Lambda.$$

Theorem

Let Λ be a finite alphabet and $\mathbf{T}$ a vector tree. Then for every finite Souslin measurable coloring of the set $\mathrm{W}^\infty(\Lambda, \mathbf{T})$ and every subspace X of $\mathrm{W}(\Lambda, \mathbf{T})$ there exists a subspace X' of $\mathrm{W}(\Lambda, \mathbf{T})$ with $X' \leq X$ such that the set $[X']^\infty_\Lambda$ is monochromatic.