A Lindelöf topological group with non-Lindelöf square

(joint work with Liuzhen Wu)

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(c) paracompactness;

(d) Lindelöfness.

It's well-known that for regular spaces, Lindelöf $\Rightarrow$ paracompact $\Rightarrow$ normal & weakly paracompact.

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Weaker version: is the square of hereditarily Lindelöf group normal or weakly paracompact?

Earlier results

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Theorem (Douwen, 1984)

There are two Lindelöf groups G and H such that $G \times H$ is not Lindelöf.

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Asume cof $(\mathcal{M}) = \omega_1$. *There is a Lindelöf group whose square is not Lindelöf.*

Theorem (Todorcevic,1993)

Assume $Pr_0(\omega_1, \omega_1, 4, \omega)$. *There is a Lindelöf group whose square is not Lindelöf.*

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Theorem (Moore, 2006)

There is an L space.

A little strengthening - L group

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The first L group appeared quite early.

Theorem (Hajnal, Juhasz, 1973)

It is consistent to have an L group.

L group

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Theorem

The group generated by Moore's L space is not Lindelöf.

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Note that for regular spaces, Lindelöf $\Rightarrow$ paracompact $\Rightarrow$ normal & weakly paracompact. So none of these 4 properties is preserved by taking square.

Combinatorial property of the osc map

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Theorem (Moore)

Let $\{\theta_\alpha : \alpha < \omega_1\}$ be a set of rationally independent reals and $\mathscr{A} \subset [\omega_1]^k$ be an uncountable family of pairwise disjoint sets, $B \in [\omega_1]^{\omega_1}$. Then for any sequence $U_i \subset (0,1)$ of open sets ($i < k$), there are $a \in \mathscr{A}$ and $\beta \in B \setminus a$ such that for any $i < k$, $frac(\theta_{a(i)} osc(a(i), \beta)) \in U_i$.

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Roughly speaking,
$\{(frac(\theta_{a(0)} osc(a(0), \beta)), ..., frac(\theta_{a(k-1)} osc(a(k-1), \beta))) : a \in \mathscr{A}, \beta \in B \setminus a\}$ is dense in $(0,1)^k$ for any appropriate $\mathscr{A}, B$. And this is the key to get the L space property.

More combinatorial properties of the osc map

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Theorem (Combinatorial property 1)

For any uncountable families of pairwise disjoint sets $\mathscr{A} \subset [\omega_1]^k$ *and* $\mathscr{B} \subset [\omega_1]^l$, *there are* $\mathscr{A}' \in [\mathscr{A}]^{\omega_1}$, $\mathscr{B}' \in [\mathscr{B}]^{\omega_1}$ *and* $\langle c_{ij} : i < k, j < l \rangle \in \mathbb{Z}^{k \times l}$ *such that for any* $a \in \mathscr{A}'$, *for any* $b \in \mathscr{B}' \setminus a$, $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$ *for any* $i < k, j < l$.

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This property allows us to refine $\mathscr{A}, \mathscr{B}$. As we are dealing with problems of the form: "for any uncountable $\mathscr{A}, \mathscr{B}$,...", combinatorial property 1 allows us dealing with the easier case: "for any uncountable $\mathscr{A}, \mathscr{B}$ with property mentioned above,...".

We also have a complement of combinatorial property 1.

Theorem (Combinatorial property 2)

For any $X \in [\omega_1]^{\omega_1}$, for any $k, l < \omega$, for any $\langle c_{ij} : i < k, j < l \rangle \in \mathbb{Z}^{k \times l}$ such that $c_{i0} = 0$ for $i < k$, there are uncountable families $\mathscr{A} \subset [X]^k$, $\mathscr{B} \subset [X]^l$ that are pairwise disjoint and for any $a \in \mathscr{A}, b \in \mathscr{B} \setminus a$, $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$ for $i < k, j < l$.

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Definition

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$$w_\beta(\alpha) = \begin{cases} f(frac(\theta_\alpha osc(\alpha, \beta) + \theta_\beta)) & : \alpha < \beta \\ 0 & : \alpha \geq \beta. \end{cases}$$

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$grp(\mathscr{L})$ – the group generated by $\mathscr{L}$ – is what we need.

Theorem

$grp(\mathscr{L})$ is an L group whose square is neither normal nor weakly paracompact.

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So none of the properties mentioned above is preserved by taking square for topological groups.

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Now, with the help of combinatorial property 1 of *osc*, we can assume that there is $\langle c_j : j < l \rangle \in \mathbb{Z}^l$ such that $osc(\alpha, b(j)) = osc(\alpha, b(0)) + c_j$ for appropriate items.

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$$\approx \sum_{j<l} n_j f(frac(x + \theta c_j + \theta^j)).$$

Using a complete accumulation point argument, θ_α and $\theta_{b(j)}$ ($j < l$) can be treated as constants. So $rang(A, \mathscr{B})$ is dense follows from Moore's Theorem that the first input $frac(\theta_\alpha osc(\alpha, b(0)))$ is dense.

Question

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Question (Arhangelskii)

Let $C_p(X)$ be Lindelöf. Is it then true that $C_p(X) \times C_p(X)$ is Lindelöf?

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Let $C_p(X)$ be Lindelöf. Is it then true that $C_p(X) \times C_p(X)$ is Lindelöf?

Question

Let X be a Banach space with weak topology w such that (X, w) is Lindelöf. Is it true that $(X, w)^2$ is Lindelöf?

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For what $n < \omega$ do we have a Lindelöf group (L group) whose n-th power is Lindelöf (L) while its $n+1$-th power is not Lindelöf?

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For what $n < \omega$ do we have a Lindelöf group (L group) whose n-th power is Lindelöf (L) while its $n+1$-th power is not Lindelöf?

The problem is that we didn't know whether there is an L space whose square is an L space.

Higher finite power and strong negative partition relation

Generalize above construction again, we get the following.

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Theorem

For any $n < \omega$, there is a topological group G such that G^n is an L group and G^{n+1} is neither normal nor weakly paracompact.

Higher finite power and strong negative partition relation

Generalize above construction again, we get the following.

Theorem

For any $n < \omega$, there is a topological group G such that G^n is an L group and G^{n+1} is neither normal nor weakly paracompact.

And this is the best we can do in ZFC.

Theorem (Kunen,1977)

Assume MA_{ω_1}. There is no space (group) X such that X^n is an L space (group) for any $n < \omega$.

Definition

(Strong coloring, Shelah) $Pr_0(\kappa, \kappa, \kappa, \sigma)$ asserts that there is a function $c : [\kappa]^2 \to \kappa$ such that whenever we are given $\gamma < \sigma$, a family $\mathscr{A} \subset [\kappa]^\gamma$ of κ many pairwise disjoint sets and a function $h : \gamma \times \gamma \to \kappa$, then there are $a < b$ in $\mathscr{A}$ such that $c(a(i), b(j)) = h(i, j)$ for any $i, j < \gamma$.

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The proof for higher finite powers of L groups actually gives us a strong negative partition relation.

Theorem

For any $n < \omega$, $Pr_0(\omega_1, \omega_1, \omega_1, n)$ *holds.*

The case for $n = 2$ is $\omega_1 \nrightarrow [\omega_1]^2_{\omega_1}$ proved by Todorcevic.

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Theorem (Shelah)

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We don't have that strong version on ω_1.

Fact

$Pr_0(\omega_1, \omega_1, \omega_1, \omega)$ is independent of ZFC.

Thank you!