Geometric singular perturbations of Poisson-Nernst-Planck systems and applications to ion channel problems

Weishi Liu
University of Kansas

Workshop on Transport of Ionic Particles in Biological Environments
Fields Institute, July 28-Aug. 1, 2014

Abstract

In this talk, we will report our work on Poisson-Nernst-Planck (PNP) type systems, a class of primitive continuum models for electrodiffusion, mainly in the content of ionic flow through membrane channels. An important modeling feature of the PNP type systems studied is the inclusion of hard-sphere potentials that account for ion size effect. We will focus on hard-sphere potentials that are ion specific. This complication is critical since ions with the same charge but different sizes could have significantly different roles in many important biological functions of living organisms. We will present an analytical framework that relies on a combination of a powerful general theory of geometric singular perturbations and of specific structures of PNP type systems. Beyond existence and uniqueness problems, we are interested in obtaining concrete characteristics of solutions that have direct implications to ionic flow properties. A particular attention is paid on effects of the ion sizes and permanent charges to electrodiffusion and ion channel functions.

What Is This ?

From Eisenberg-L. 07 SIMA:

$$z_1 c_1^a e^{z_1(\phi^a - \phi^{a,m})} + z_2 c_2^a e^{z_2(\phi^a - \phi^{a,m})} + Q = 0,$$

$$z_1 c_1^b e^{z_1(\phi^b - \phi^{b,m})} + z_2 c_2^b e^{z_2(\phi^b - \phi^{b,m})} + Q = 0,$$

$$\frac{z_2 - z_1}{z_2} c_1^{a,l} = c_1^a e^{z_1(\phi^a - \phi^{a,m})} + c_2^a e^{z_2(\phi^a - \phi^{a,m})} + Q(\phi^a - \phi^{a,m}),$$

$$\frac{z_2 - z_1}{z_2} c_1^{b,r} = c_1^b e^{z_1(\phi^b - \phi^{b,m})} + c_2^b e^{z_2(\phi^b - \phi^{b,m})} + Q(\phi^b - \phi^{b,m}),$$

$$J_1 = \frac{c_1^L - c_1^{a,l}}{H(a)}\left(1 + \frac{z_1(\phi^L - \phi^{a,l})}{\ln c_1^L - \ln c_1^{a,l}}\right) = \frac{c_1^{b,r} - c_1^R}{H(1) - H(b)}\left(1 + \frac{z_1(\phi^{b,r} - \phi^R)}{\ln c_1^{b,r} - \ln c_1^R}\right),$$

$$J_2 = \frac{c_2^L - c_2^{a,l}}{H(a)}\left(1 + \frac{z_2(\phi^L - \phi^{a,l})}{\ln c_2^L - \ln c_2^{a,l}}\right) = \frac{c_2^{b,r} - c_2^R}{H(1) - H(b)}\left(1 + \frac{z_2(\phi^{b,r} - \phi^R)}{\ln c_2^{b,r} - \ln c_2^R}\right),$$

$$\phi^{b,m} = \phi^{a,m} - (z_1 J_1 + z_2 J_2)y,$$

$$c_1^{b,m} = e^{z_1 z_2 (J_1 + J_2)y} c_1^{a,m} - \frac{Q J_1}{z_1(J_1 + J_2)}\left(1 - e^{z_1 z_2 (J_1 + J_2)y}\right),$$

$$J_1 + J_2 = -\frac{(z_1 - z_2)(c_1^{a,m} - c_1^{b,m}) + z_2 Q(\phi^{a,m} - \phi^{b,m})}{z_2(H(b) - H(a))}.$$

Outline

Part I: Geometric singular perturbation theory (GSP).

Part II: GSP for PNP:

General framework of GSP + Special structures of PNP.

Part III: Effects of ion size, permanent charge, and channel geometry.

Part I: Quick Review of GSP

1. Invariant manifold theory: normally hyperbolic and center manifolds.

More than one century of history involving works of first rate mathematicians.

2. A general dynamical system framework of GSP (last thirty years or so).

– Slow and fast systems, their limits, and their general relations

– Slow manifolds, normal hyperbolicity and turning points

3. Specifics in GSP

– Exchange Lemmas for NH slow manifolds

– Dynamics for slow manifolds with turning points

  * Exchange Lemmas for slow manifolds with stability-loss turning points
  * Tent structure for slow manifolds with stability-gain turning points

1. Invariant manifold theory for nonlinear dynamics

Consider $x' = f(x)$, $x \in \mathbb{R}^n$, with flow ϕ^t (w/o explicit formula).

Let Y be a compact invariant manifold: $\phi^t(Y) = Y$. Dynamics near Y ?

The simplest invariant manifold is an equilibrium; say, $Y = \{0\}$.

E-values and e-vectors of $Df(0)$ determine dynamics of $X' = Df(0)X$.

Does the dynamics near 0 persist for the nonlinear equation? How to present?

Invariant manifold theory: from linear to nonlinear.

For general invariant manifold Y:

- $\forall y \in Y$, the linearization along $y \cdot t = \phi^t(y) \in Y$ is $X' = D_x f(y \cdot t)X$.

Let $\Phi(y,t) = D_x\phi^t(y) : T_y\mathbb{R}^n \to T_{y\cdot t}\mathbb{R}^n$ be its principal FMS: $\Phi(y,0) = I$.

– If Y is a smooth manifold, then $\Phi(y,t) : T_yY \to T_{y\cdot t}Y$.

– If (i) $\exists\ \Phi(\cdot,t)$-invariant splitting of $T_Y\mathbb{R}^n = \cup_{y\in Y} T_y\mathbb{R}^n$:

$$T_y\mathbb{R}^n = V^s(y) \oplus T_yY \oplus V^u(y) \text{ and } \Phi(y,t) : V^{s,u}(y) \to V^{s,u}(y\cdot t);$$

(ii) the generalized Lyapunov numbers conditions hold:

$$\alpha = \limsup_{t\to\infty} \frac{1}{t} \ln \|\Phi_s(y,t)\| < 0, \quad \beta = \liminf_{t\to-\infty} \frac{1}{t} \ln \|\Phi_u(y,t)\| > 0.$$

$$\gamma = \limsup_{t\to\infty} \frac{\ln \|\Phi_s(y,t)\|}{\ln \|\Phi_c(y\cdot t,-t)\|^{-1}} < \frac{1}{k}, \quad \sigma = \limsup_{t\to-\infty} \frac{\ln \|\Phi_u(y,t)\|}{\ln \|\Phi_c(y\cdot t,-t)\|^{-1}} < \frac{1}{k},$$

then Y is k-normally hyperbolic (NH).

Consequences of NH: [Fenichel 71 & Hirsch-Pugh-Shub 76]

(1) Stable and unstable manifolds with invariant foliations:

$\exists\, W^s(Y) = \cup_{y\in Y} W^s(y),\ W^u(Y) = \cup_{y\in Y} W^u(y);\ \phi^t(W^{s,u}(y)) \subset W^{s,u}(y\cdot t),$

$|\phi^t(x_2) - \phi^t(x_1)| \le K e^{(\alpha+\delta)t} |x_2 - x_1|$ for $x_j \in W^s(y)$ for $t > 0$,

$|\phi^t(x_2) - \phi^t(x_1)| \le K e^{(\beta-\delta)t} |x_2 - x_1|$ for $x_j \in W^u(y)$ for $t < 0$.

(2) Forward bounded orbits near Y lie in $W^s(Y)$,

Backward bounded orbits near Y lie in $W^u(Y)$,

Bounded orbits near Y lie in Y. (Interesting dynamics near Y occurs on Y.)

(3) The above persists under perturbations. (Bifurcations near Y occur on Y.)

*If, either (ii) does not hold or Y is not a mfld, then Y will NOT be NH, and will NOT persist [Mane 77]. One needs to replace T_yY by a generalized tangent space $V^c(y)$ with (i) and (ii). Then, there is a center manifold $W^c(Y)$ so that $Y \subset W^c(Y)$ and $W^c(Y)$ is NH, and hence, (1), (2), (3) hold for $W^c(Y)$ replacing Y.

[Chow-L.-Yi: AMS Trans. 2000 & JDE 2000]

2. Dynamical system framework for GSP

2.1. A standard form of singularly perturbed problems: slow and fast systems

$$(S)_\varepsilon \quad \begin{cases} \varepsilon\dot{x}(\tau) = f(x,y;\varepsilon), \\ \dot{y}(\tau) = g(x,y;\varepsilon). \end{cases} \iff (F)_\varepsilon \quad \begin{cases} x'(t) = f(x,y;\varepsilon), \\ y'(t) = \varepsilon g(x,y;\varepsilon). \end{cases}$$

2.2. Limiting slow and fast systems, slow manifolds, and general considerations

$$(S)_0 \quad \begin{cases} 0 = f(x,y;0), \\ \dot{y} = g(x,y;0). \end{cases} \qquad (F)_0 \quad \begin{cases} x' = f(x,y;0), \\ y' = 0. \end{cases}$$

$$\text{Slow manifold: } \mathcal{Z}_0 = \{(x,y) : f(x,y;0) = 0\} = \{x = h_0(y)\}.$$

- $(S)_0$ on $\mathcal{Z}_0$: $\dot{y} = g(h_0(y), y; 0)$ gives limiting dynamics of slow variable y.
- $(F)_0$ determines limiting dynamics of fast variable x parameterized by y.

How to lift limiting slow and fast information to $\varepsilon > 0$ small ?

2.3. Normally hyperbolic slow manifolds

- Slow manifold $\mathcal{Z}_0$ is a set of equilibria of $(F)_0$.

- Linearization at each $(h_0(y), y) \in \mathcal{Z}_0$ gives

$$\begin{pmatrix} f_x(h_0(y), y) & f_y(h_0(y), y) \\ 0 & 0 \end{pmatrix},$$

which has $\dim \mathcal{Z}_0$ many zero e-values and others are those of $f_x(h_0(y), y)$.

- If all e-values of $f_x(h_0(y), y)$ have non-zero real parts, then $\mathcal{Z}_0$ is NH.

  One can then apply NH theory to study the dynamics of $(F)_\varepsilon$ near $\mathcal{Z}_0$.

  Particularly, $\exists\, h_\varepsilon(y) \sim h_0(y)$ so that $\mathcal{Z}_\varepsilon = \{x = h_\varepsilon(y)\}$ is invariant for small ε.

  On $\mathcal{Z}_\varepsilon$, $\dot{y} = g(h_\varepsilon(y), y; \varepsilon)$, which governs that near $\mathcal{Z}_\varepsilon$ through persistence.

2.4. Turning points

- If, for some y_*, $f_x(h_0(y_*), y_*)$ has e-values with zero real parts, then $(h_0(y_*), y_*)$ is called a turning point where NH of $\mathcal{Z}_0$ is lost.

  Folding points of $\mathcal{Z}_0$ are also turning points.

  Generically, the set of turning points T form a co-dim-one sub-manifold of $\mathcal{Z}_0$.

- A rough classification:

  (i) Stability-loss: as slow flow moves from one side of T to the other, real part of an (fast) e-value changes from negative (stable) to positive (unstable).

  A key feature: Canard points, Delay of stability-loss and its sensitivity.

  (ii) Stability-gain: as slow flow moves from one side of T to the other, real part of an (fast) e-value changes from positive (unstable) to negative (stable).

  A Key feature: Tent structure and its robustness.

  (iii) Others $\cdots$

3. Specifics in GSP

3.1. Exchange Lemmas for NH slow manifolds [C. Jones: Lect. Note in Math]

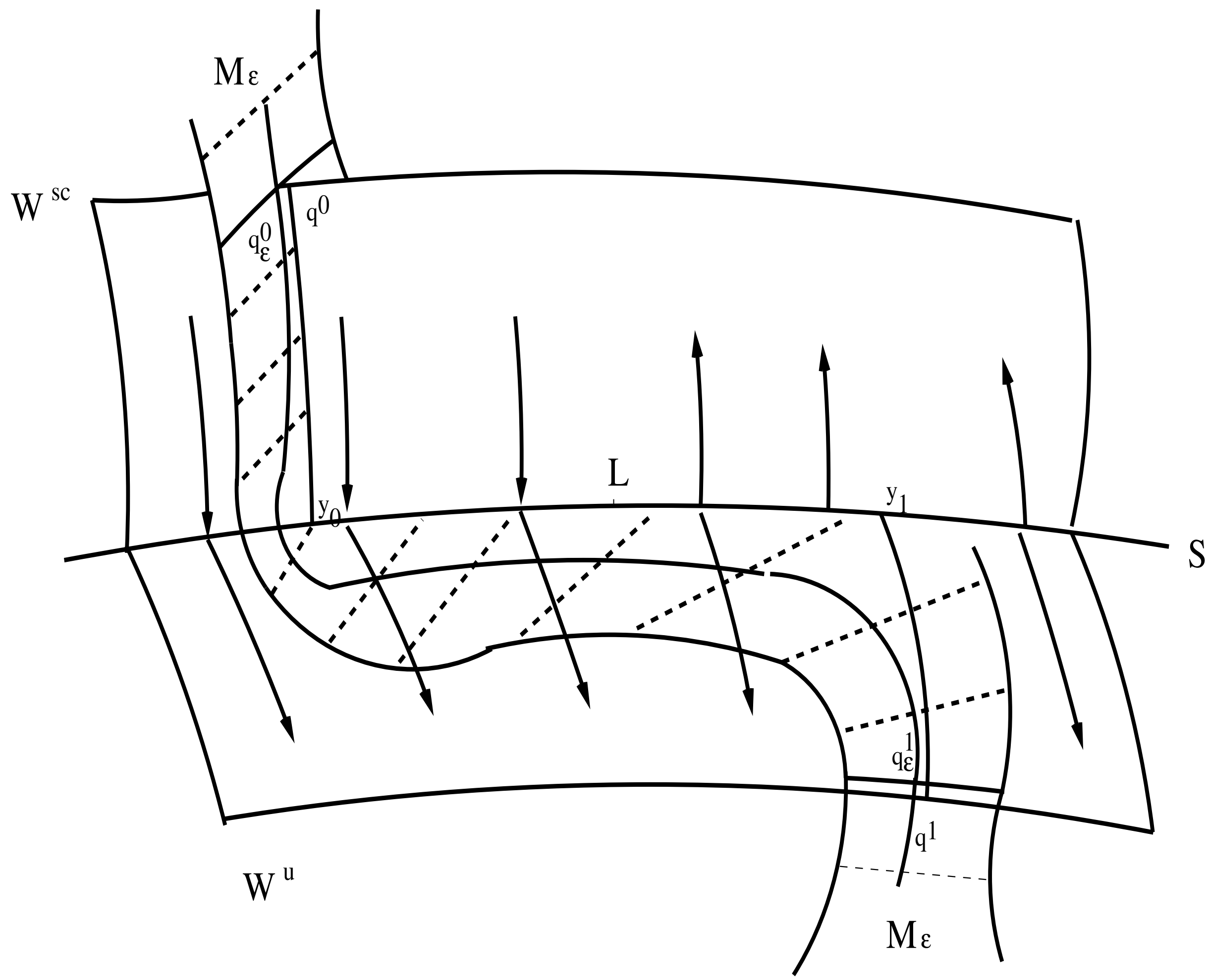

Figure 1: Entry configuration of M_0 gives exit configuration of M_ε

3.2. Exchange Lemmas for stability-loss turning points [Liu 2000 JDE]

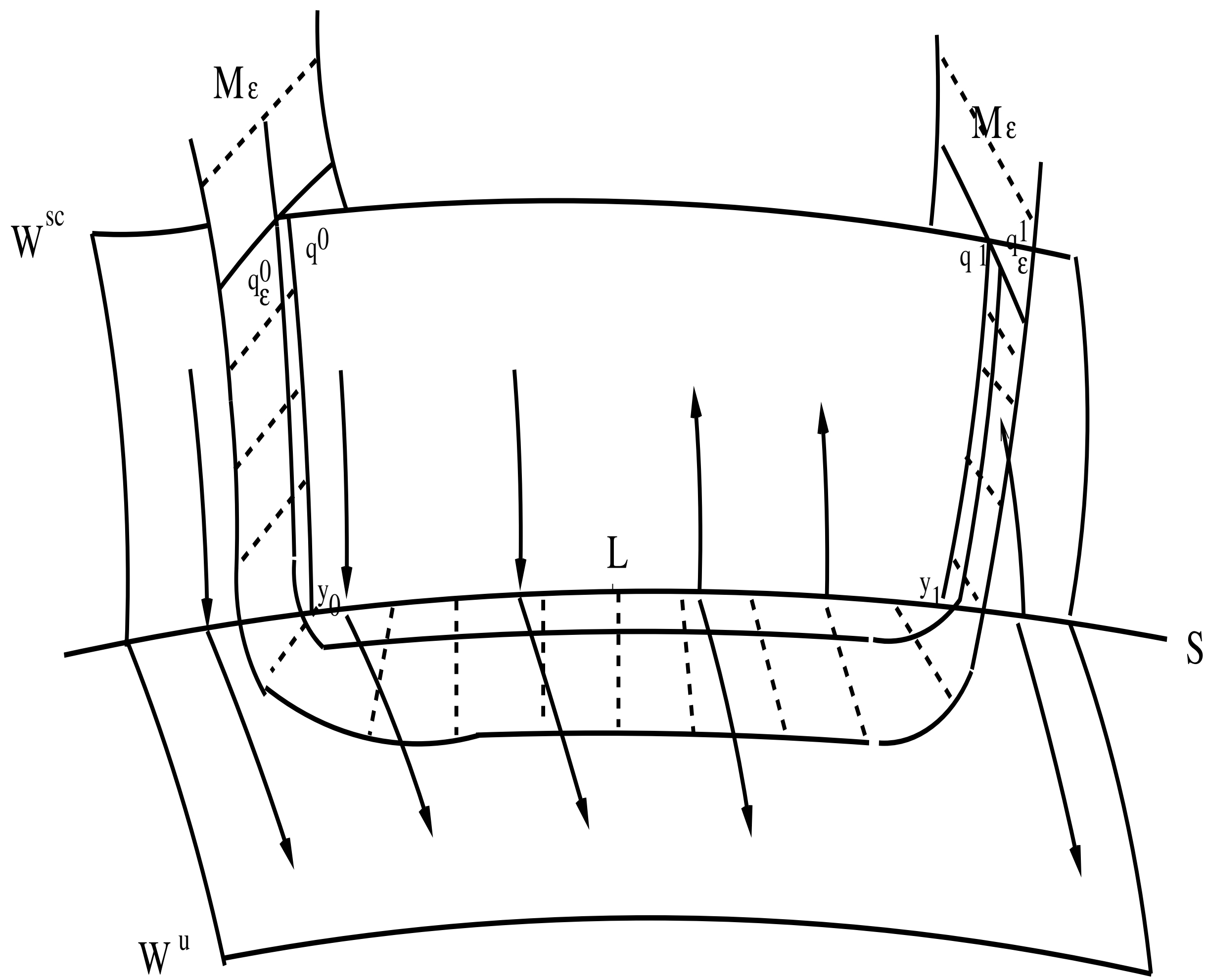

Figure 2: Delay of stability-loss

3.3. Tent structure for stability-gain turning points [Liu TBA]

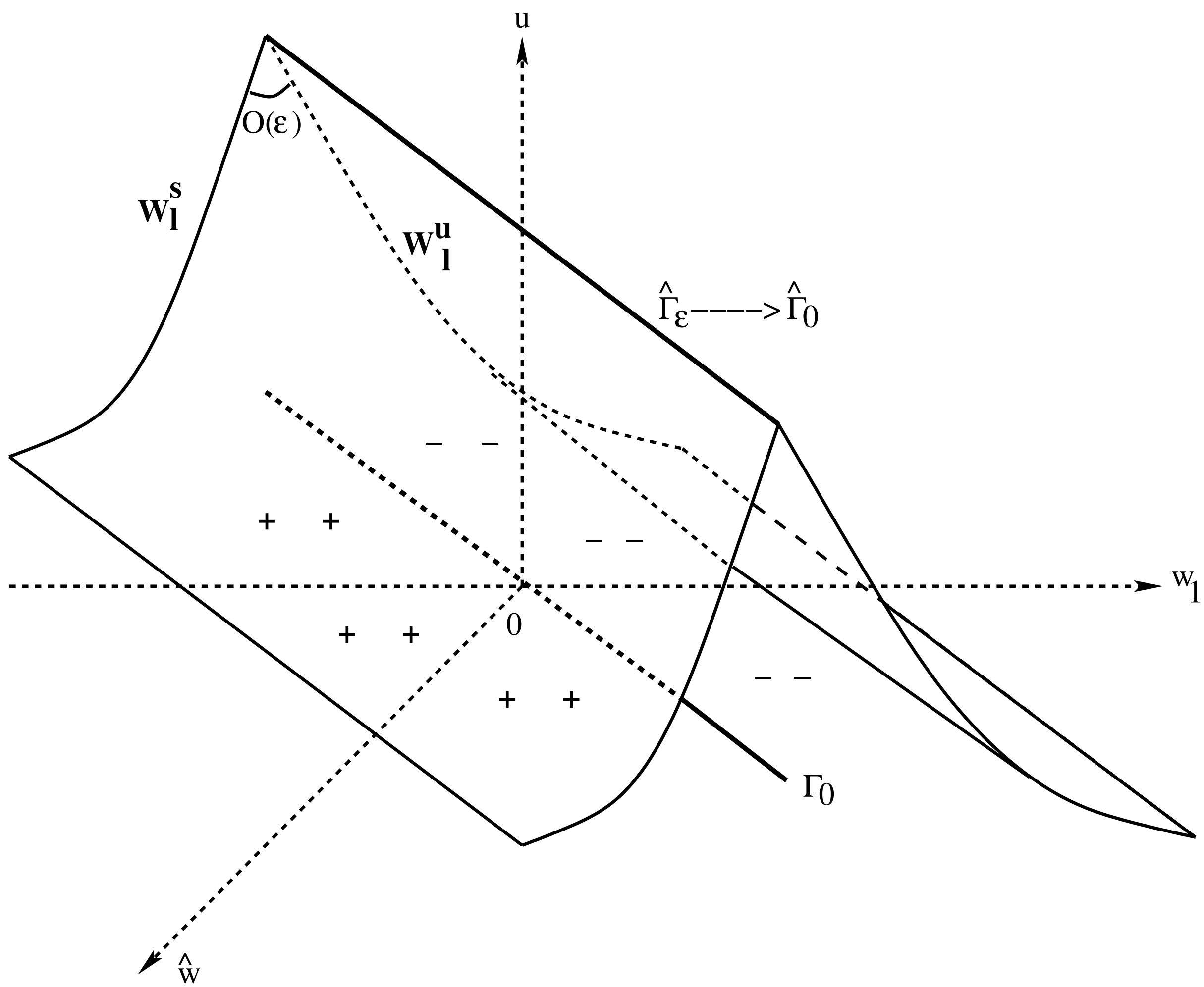

Figure 3: Tent structure, its roughness, and nearby dynamics

Part II: GSP for PNP

1. General framework of GSP

2. Special structures of PNP (most important ingredients for concrete information)

3. Matching: yields (local) double-layers and brings (global) BV into picture

4. Governing systems for singular orbits of BVP of PNP

1. Ion channel, ionic flow, PNP model

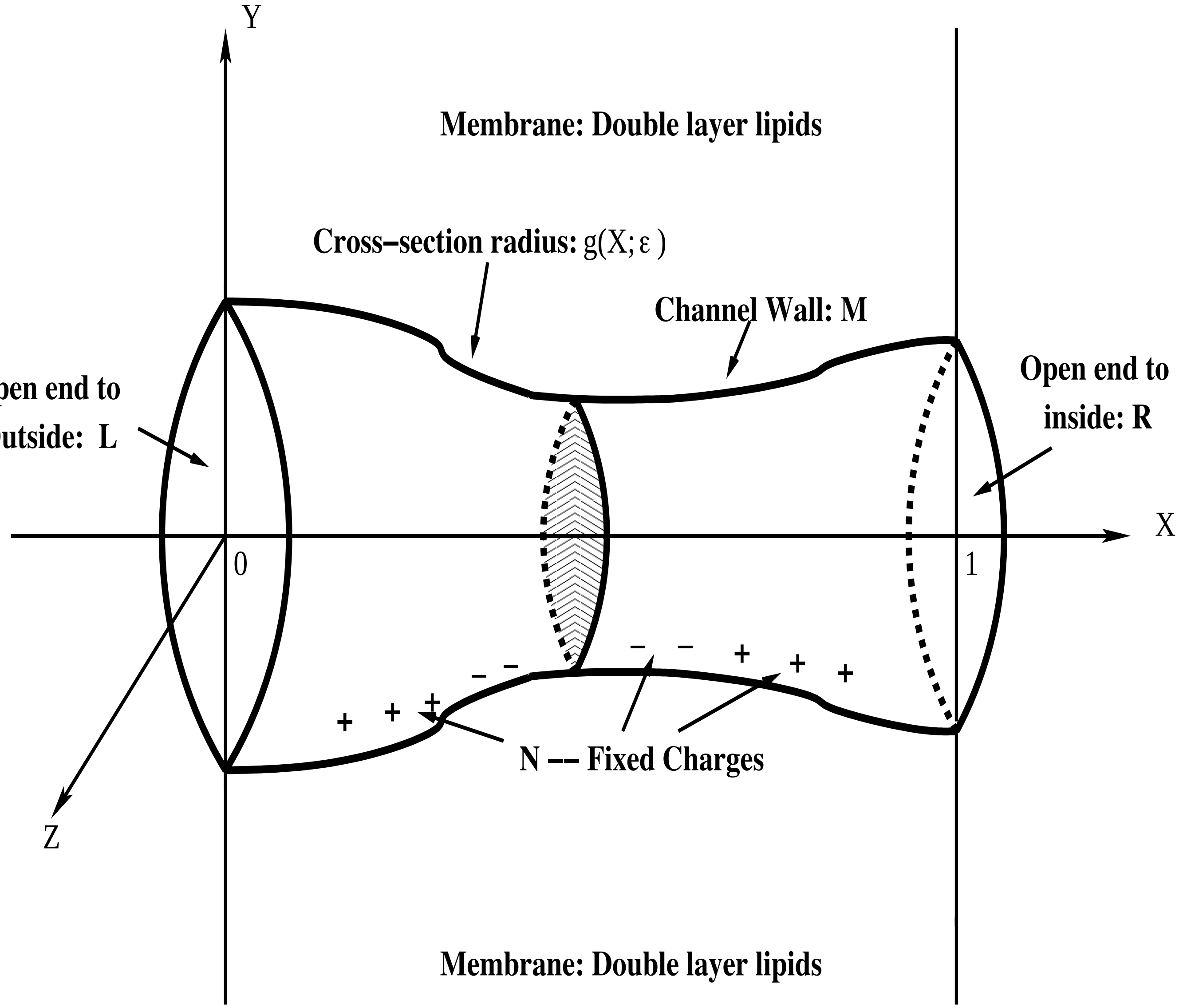

Figure 4: What Are Ion Channels: Shape and Permanent Charge

Quasi-one-dim PNP model for ionic flows

Poisson: $$\frac{1}{A(x)}\frac{d}{dx}\Big(\epsilon^2 A(x)\frac{d\phi}{dx}\Big) = -e\sum z_s c_s - Q(x),$$

Nernst-Planck: $$-J_j = \frac{1}{k_BT}D_jA(x)c_j\frac{d\mu_j}{dx}, \quad \frac{dJ_j}{dx} = 0.$$

BV: $\phi(0) = \mathcal{V}$, $c_j(0) = L_j$; $\phi(1) = 0$, $c_j(1) = R_j$.

c_j – concentration, J_j – flux density, z_j – valence, D_j – diffusion constant,

ϕ–electric potential, ϵ^2–dielectric, $A(x)$–area over x, $Q(x)$–permanent charge

Electrochemical potential: $\mu_j(\phi, \{c_i\}) = \mu_j^{id} + \mu_j^{ex}$:

Ideal component $\mu_j^{id} = z_je\phi + k_BT\ln c_j$; Excess potential μ_j^{ex} for ion size.

Current-Voltage (I-V) relation: $\mathcal{I} = \sum z_jJ_j(\mathcal{V}; L, R)$.

2. GSP for cPNP w/ piecewise constant $Q(x)$: [Liu 09 JDE]

2.1. Reformulate BVP to a connecting problem (after a rescaling)

Introduce $u = \varepsilon\dot{\phi}$ and $w = x$. cPNP becomes, for $k = 1, 2, \cdots, n$,

$$\varepsilon\dot{\phi} = u, \quad \varepsilon\dot{u} = -\sum_{s=1}^{n} z_s c_s - Q(w) - \varepsilon\frac{A'(w)}{A(w)}u,$$
$$\varepsilon\dot{c}_k = -z_k c_k u - \varepsilon J_k A^{-1}(w), \quad \dot{J} = 0, \quad \dot{w} = 1.$$

Associated to boundary conditions, introduce

$$B_L = \{(\phi, u, C, J, w) \in \mathbb{R}^{2n+3} : \phi = \mathcal{V},\ C = L,\ w = 0\},$$
$$B_R = \{(\phi, u, C, J, w) \in \mathbb{R}^{2n+3} : \phi = 0,\ C = R,\ w = 1\}.$$

BVP $\Longleftrightarrow$ A connecting orbit from B_L to B_R.

Let M_L^ε be the collection of all forward orbits starting from B_L and M_R^ε be the collection of all backward orbits starting from B_R.

Then, for $\epsilon > 0$ small, the vector field is not tangent to B_L and B_R.

$$\dim B_L = \dim B_R = n + 1 \Longrightarrow \dim M_L^\epsilon = \dim M_R^\epsilon = n + 2.$$

Generically, M_L^ϵ and M_R^ϵ intersect transversally. In this case,

$$\dim(M_L^\epsilon \cap M_R^\epsilon) = \dim M_L^\epsilon + \dim M_R^\epsilon - \dim \mathbb{R}^{2n+3} = 1.$$

It suffices to show that M_L^ϵ and M_R^ϵ indeed intersect transversally. The idea is:

(i) to construct a singular orbit: union of fast and slow orbits of different limiting systems, where fast orbits represent the boundary/internal layers and slow orbits connect the boundary/internal layers;

(ii) to examine the evolutions of M_L^ϵ and M_R^ϵ along the singular orbit for transversality and apply the Exchange Lemma.

2.2. Construction of Singular Orbits over $[0,1]$.

Pre-assign the values of ϕ, c_k's at jump point x_j of $Q(x)$ for $j = 1, 2, \cdots, m-1$,

$$\phi(x_j) = \phi^{[j]}, \quad c_k(x_j) = c_k^{[j]}, \quad k = 1, 2, \cdots, n \tag{1}$$

with given $\phi^{[0]} = \mathcal{V}$ and $c_k^{[0]} = L_k$ at $x_0 = 0$, $\phi^{[m]} = 0$ and $c_k^{[m]} = R_k$ at $x_m = 1$, and introduce the set, for $j = 0, 1, \cdots, m$,

$$B_j = \{(\phi, u, C, J, w):\ \phi = \phi^{[j]}, C = C^{[j]}, w = x_j\}. \tag{2}$$

Two main steps for a construction of a singular orbits over $[0,1]$

- Singular orbits on $[x_{j-1}, x_j]$ between B_{j-1} and B_j with $Q(x) = Q_j$.
- Matching them at jump points $x = x_j$'s to form a singular orbit on $[0,1]$.

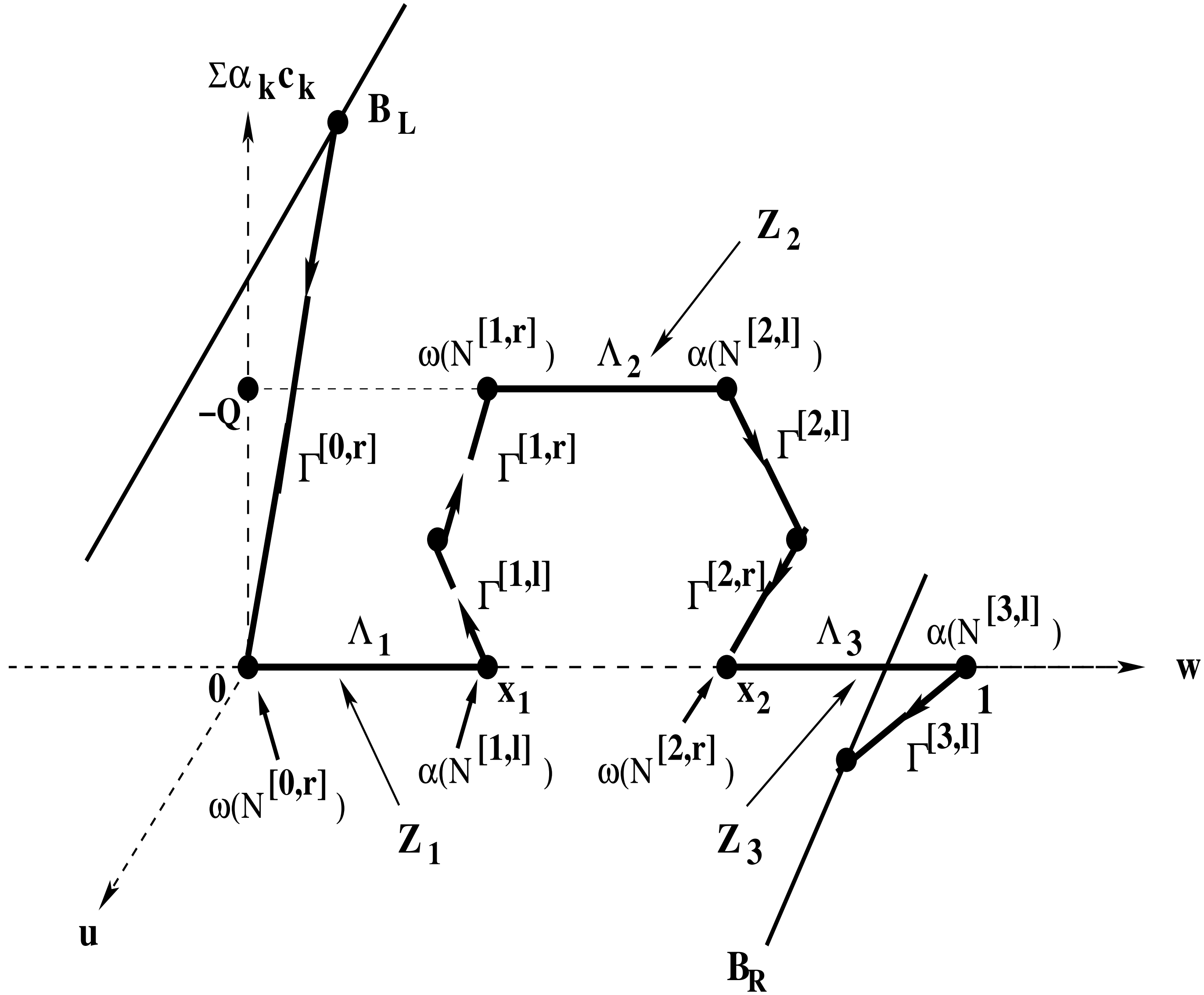

Figure 5: A singular orbit: Double-Layer at each x_j

2.2.1. Singular orbit over $[x_{j-1}, x_j]$ between B_{j-1} and B_j with $Q(x) = Q_j$.

Each such an orbit will consist of two singular layers $\Gamma^{[j-1,r]}$ at $x = x_{j-1}$, and $\Gamma^{[j,l]}$ at $x = x_j$, and a regular layer Λ_j over the interval $[x_{j-1}, x_j]$.

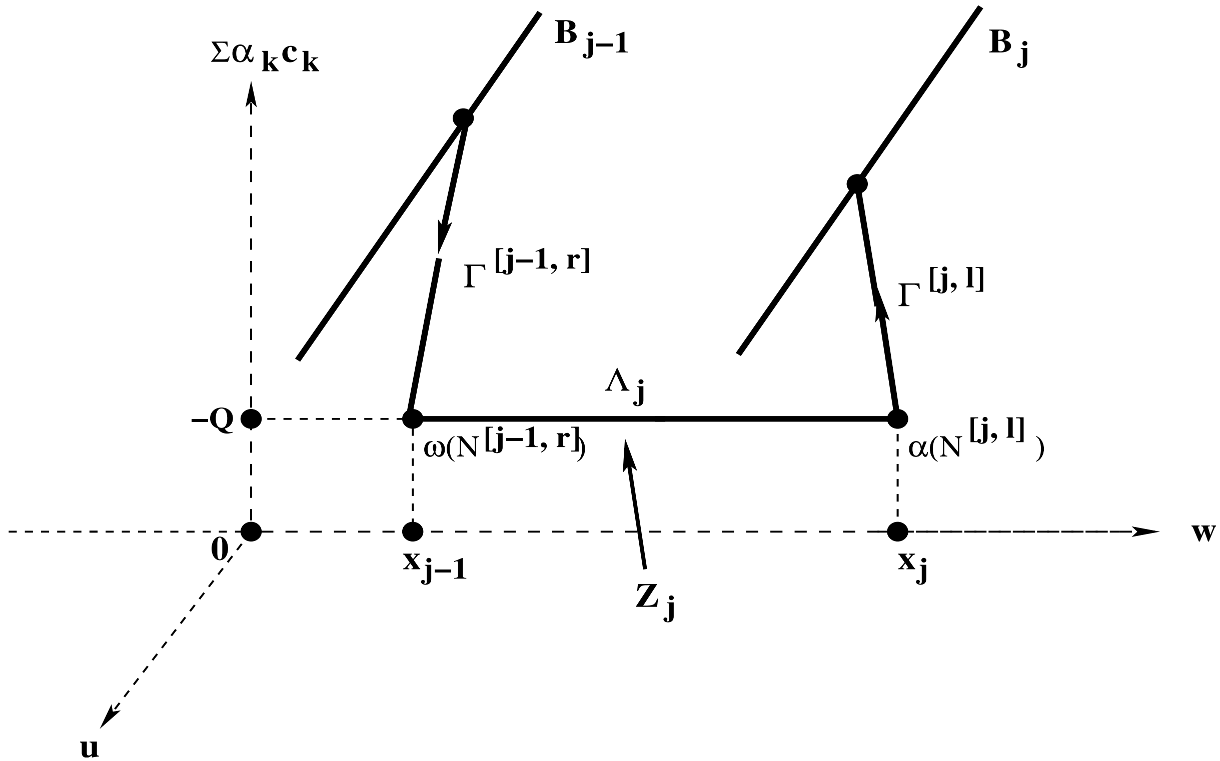

Figure 6: Singular orbit over $[x_{j-1}, x_j]$

– Fast dynamics and singular layers.

The slow manifold is $\mathcal{Z}_j = \{u = 0,\ \sum_{s=1}^{n} z_s c_s + Q_j = 0\}$.

Note that $\dim \mathcal{Z}_j = 2n + 1$ – co-dim two.

In terms of the independent variable $\xi = x/\epsilon$, we obtain the fast system,

$$\phi' = u, \quad u' = -\sum_{s=1}^{n} z_s c_s - Q_j - \varepsilon \frac{A'(w)}{A(w)} u,$$
$$c_k' = -z_k c_k u - \varepsilon J_k A^{-1}(w), \quad J' = 0, \quad w' = \varepsilon.$$

The limiting fast system is, for $k = 1, 2, \cdots, n$,

$$\phi' = u, \quad u' = -\sum_{s=1}^{n} z_s c_s - Q_j,$$
$$c_k' = -z_k c_k u, \quad J' = 0, \quad w' = 0.$$

Two e-values normal to $\mathcal{Z}_j$ are $\pm\sqrt{\sum z_s^2 c_s}$ (Debye length). Thus, $\mathcal{Z}_j$ is NH.

Special structure of the limiting fast system:

Proposition 1. *The limiting fast system has a complete set of* $(2n+2)$ *first integrals given by, for* $k = 1, 2, \cdots, n$,

$$G_k = \ln c_k + z_k \phi, \quad G_{n+1} = \frac{1}{2}u^2 - \sum_{s=1}^{n} c_s + Q_j \phi,$$

$$G_{n+1+k} = J_k \ \text{ and } \ G_{2n+2} = w.$$

Consequences:

One can determine $u^{[j-1,+]}$ and $u^{[j,-]}$, and $\omega(B_{j-1})$ and $\alpha(B_j)$ up to J.

– Slow dynamics to connect $\omega(B_{j-1})$ and $\alpha(B_j)$.

Introduce $u = \varepsilon p, \quad z_n c_n = -\sum_{s=1}^{n-1} z_s c_s - Q_j - \varepsilon q$.

In replacing u with p and c_n with q, slow system becomes, for $k = 1, \cdots, n-1$,

$$\dot{\phi} = p, \quad \varepsilon \dot{p} = q - \varepsilon \frac{A'(w)}{A(w)} p,$$

$$\varepsilon \dot{q} = \Big(\sum_{s=1}^{n-1} (z_s - z_n) z_s c_s - z_n Q_j - \varepsilon z_n q \Big) p + A^{-1}(w) \sum_{s=1}^{n} z_s J_s,$$

$$\dot{c}_k = -z_k p c_k - J_k A^{-1}(w), \quad \dot{J} = 0, \quad \dot{w} = 1.$$

When $\varepsilon = 0$, it is

$$\dot{\phi} = p, \quad 0 = q,$$

$$0 = \Big(\sum_{s=1}^{n-1} (z_s - z_n) z_s c_s - z_n Q_j \Big) p + A^{-1}(w) \sum_{s=1}^{n} z_s J_s,$$

$$\dot{c}_k = -z_k p c_k - J_k A^{-1}(w), \quad \dot{J} = 0, \quad \dot{w} = 1.$$

For this system, the slow manifold is

$$\mathcal{S}_j = \left\{ p = -\frac{A^{-1}(w)\sum_{s=1}^{n} z_s J_s}{\sum_{s=1}^{n-1}(z_s - z_n)z_s c_s - z_n Q_j},\ q = 0 \right\}.$$

The limiting slow dynamics on $\mathcal{S}_j$ is, with $\mathcal{I} = \sum_{s=1}^{n} z_s J_s$,

$$\begin{aligned}
\dot{\phi} &= -\frac{A^{-1}(w)\mathcal{I}}{\sum_{s=1}^{n-1}(z_s - z_n)z_s c_s - z_n Q_j},\\
\dot{c}_k &= \frac{A^{-1}(w)\mathcal{I}}{\sum_{s=1}^{n-1}(z_s - z_n)z_s c_s - z_n Q_j} z_k c_k - A^{-1}(w) J_k,\\
\dot{J} &= 0, \quad \dot{w} = 1.
\end{aligned}$$

A crucial observation is that, on $\mathcal{S}_j$ where $q = 0$,

$$\sum_{s=1}^{n-1} z_s c_s + Q_j = -z_n c_n \Longrightarrow \sum_{s=1}^{n-1}(z_s - z_n)z_s c_s - z_n Q_j = \sum_{s=1}^{n} z_s^2 c_s.$$

Multiply $A(w)\sum_{s=1}^{n} z_s^2 c_s$ on the RHS to get

$$\frac{d}{d\tau}\phi = -\mathcal{I}, \quad \frac{d}{d\tau}C = D(J)C, \quad \sum_{s=1}^{n} z_s c_s + Q_j = 0,$$

$$\frac{d}{d\tau}J = 0, \quad \frac{d}{d\tau}w = A(w)\sum_{s=1}^{n} z_s^2 c_s,$$

where $D(J) = \Gamma - Jb^T$ with $\Gamma = \mathcal{I}$ diag$(z_1, \cdots, z_n)$ and $b^T = (z_1^2, \cdots, z_n^2)$.

Solving this system from $\omega(B_{j-1})$ to $\alpha(B_j)$, one gets $J^{[j]}$ over $[x_{j-1}, x_j]$.

2.2.2. Global matching: $u^{[j,-]} = u^{[j,+]}$ and $J^{[1]} = J^{[2]} = \cdots = J^{[m]}$.

$m - 1 + n(m-1) = (n+1)(m-1) =$ the number of preassigned unknowns.

The result gives the governing system for singular orbits of the BVP.

2.3. Exchange Lemma allows one to lift the singular orbit to a true orbit.

Part III

Effects of ion size, permanent charge and channel geometry

1. Ion size effects: PNP w/ Hard-Sphere (HS) potentials

– Critical potentials for ion size balance and ion preference (selectivity)

– Scaling Laws in boundary concentrations

2. Effects of small permanent charge and channel geometry

1. PNP with hard-sphere potentials (HS)

1.1. Why do we care about ion sizes

Serious weakness of cPNP: treating $Na^+ = K^+$

In real world, $Na^+ \neq K^+$ significantly

Key difference: $Na^+ < K^+$ in ion size

Excess potential μ_i^{ex} accounts for finite size of ions

to distinguish ions with same valence but different sizes.

1.2. A one-dim non-local HS potential μ_j^{HS}

Percus-Yevick (70s) and Rosenfeld (93) model: (exact)

$$\mu_j^{HS} = \frac{\delta\Omega(\{c_i\})}{\delta c_j},$$

where

$$\Omega(\{c_i\}) = -\int n_0(x;\{c_i\}) \ln(1 - n_1(x;\{c_i\}))dx,$$

$$n_l(x;\{c_i\}) = \sum_i \int c_i(x')\omega_l^i(x-x')dx', \quad l = 0, 1$$

$$\omega_0^i(x) = \frac{\delta(x-r_i)+\delta(x+r_i)}{2}, \quad \omega_1^i(x) = \Theta(r_i - |x|).$$

δ: Dirac function; Θ: Heaviside function; r_i: radius of ith ions.

1.3. A local HS potential μ_j^{LHS} for 3-dim

Bikerman's model (42):

$$\mu_j^{LHS}(x) = -\ln\Big(1 - \frac{4\pi}{3}\sum_i r_i^3 c_i(x)\Big) - \text{not ion specific.}$$

Many refined models $\cdots\cdots$

Boublik-Mansoori-Carnahan-Starling-Leland model (70-71):

Very accurate and more sophisticate, up to lowest order in radii,

$$\mu_j^{LHS}(x) = 8\sum_i (r_j + r_i)^3 c_i(x) + O(r^6) - \text{ion specific.}$$

1.4. Effects of ion size [Ji-L: JDDE 12, Tu-Zhang-L: JDDE 12, Lin-L.-Yi-Zhang: SIADS 13]

How ion sizes affect I-V relation:
bdry potential V and/or bdry concentrations L_j, R_j.

Upshots: Allow one to obtain "explicit" information for direct applications.

It's like you have the quadratic formula to study how roots depend on coefficients.

Electroneutrality, $n=2$: $L := z_1 L_1 = -z_2 L_2$, $R := z_1 R_1 = -z_2 R_2$

Theorem 1. [I-V relation] *Let* $r = r_1$ *and* $\lambda = r_2/r_1$.

Let $I(V;\varepsilon,r) = I_0(V;\varepsilon) + I_1(V;\lambda,\varepsilon)r + o(r)$. *Then,*

$$I_0(V;0) = (D_1 - D_2)(L-R) + \frac{(z_1D_1 - z_2D_2)(L-R)}{\ln L - \ln R}V,$$

$$\begin{aligned} I_1(V;\lambda,0) = & \frac{2(z_1D_1 - z_2D_2)(\lambda-1)(L-R)^2}{z_1z_2(\ln L - \ln R)} \\ & - \frac{2(D_1-D_2)(z_1\lambda - z_2)(L^2-R^2)}{z_1z_2} \\ & - \frac{2(z_1D_1 - z_2D_2)(z_1\lambda - z_2)[(L^2-R^2)(\ln L - \ln R) - 2(L-R)^2]}{z_1z_2(\ln L - \ln R)^2}V. \end{aligned}$$

Two critical voltages for ion size effects: V_c and V^c

Let V_c be such that $I_1(V_c;\lambda) = 0$:

$$V_c = \frac{(\lambda-1)(L-R)(\ln L - \ln R)}{(z_1\lambda - z_2)((L+R)(\ln L - \ln R) - 2(L-R))} - \frac{(D_1-D_2)(L+R)(\ln L - \ln R)^2}{(z_1 D_1 - z_2 D_2)((L+R)(\ln L - \ln R) - 2(L-R))}.$$

Let V^c be such that $\partial_\lambda I_1(V^c;\lambda) = 0$:

$$V^c = \frac{(L-R)(\ln L - \ln R)}{z_1((L+R)(\ln L - \ln R) - 2(L-R))} - \frac{(D_1-D_2)(L+R)(\ln L - \ln R)^2}{(z_1 D_1 - z_2 D_2)((L+R)(\ln L - \ln R) - 2(L-R))}.$$

Theorem 2. [Size-Balance-Voltage V_c]

For $\varepsilon > 0$ small and $r > 0$ small,

(i) if $V > V_c$, then ion sizes enhance *current: $I(V;\varepsilon,r) > I(V;\varepsilon,0)$;*

(ii) if $V < V_c$, then ion sizes reduce *current: $I(V;\varepsilon,r) < I(V;\varepsilon,0)$.*

Theorem 3. [Size-Selectivity-Voltage V^c]

For $\varepsilon > 0$ small and $r > 0$ small,

(i) if $V > V^c$, the current I is increasing *in λ*

*(*smaller *positive ion species is '*preferred*', say, Na^+ over K^+);*

(ii) if $V < V^c$, the current I is decreasing *in λ*

*(*larger *positive ion species is '*preferred*', say, K^+ over Na^+).*

Electrodiffusive Contribution to Selectivity ?

Scaling Properties in Boundary Concentrations

Recall $I(V;\varepsilon,r) = I_0(V;\varepsilon) + I_1(V;\varepsilon)r + o(r)$.

$I_0(V;0) = I_0(V;L_j,R_j)$ – point-charge component,

$I_1(V;0) = I_1(V;L_j,R_j)$ – ion size component,

$V_c = V_c(L_j,R_j), \quad V^c = V^c(L_j,R_j)$ – two critical voltages.

Theorem 4. [Scaling Laws in Bdry Concentrations]

(i) I_0 *scales* linearly *in* (L_j,R_j)*:* $I_0(V;\sigma L_j,\sigma R_j) = \sigma I_0(V;L_j,R_j)$*;*

(ii) I_1 *scales* quadratically *in* (L_j,R_j)*:* $I_1(V;\sigma L_j,\sigma R_j) = \sigma^2 I_1(V;L_j,R_j)$*;*

(iii) V_c *and* V^c *scale* invariantly *in* (L_j,R_j)*:* $V_c^c(\sigma L_j,\sigma R_j) = V_c^c(L_j,R_j)$.

2. Effects of permanent charge and channel geometry via cPNP

2.1. Reversal charge **and** reversal potential:

– Signs of fluxes J_k's vs sign of current $\mathcal{I}$

– Not so intuitive properties

2.2. Effects of permanent charges **and** channel geometry

– Permanent charge effects on fluxes

– Effects of channel geometry with and without permanent charges

2.1 Reversal charge and potential: Eisenberg-L.-Xu (Submitted)

- From NP: $J_k \int_0^1 \frac{k_B T}{A(x) D_k c_k(x)} dx = \mu_k(0) - \mu_k(1).$

The sign of J_k is determined by bdry electrochemical potentials.

Permanent charges cannot do anything about the sign of J_k BUT

do affect its magnitude. Hope: Chang the sign of $\mathcal{I} = \sum z_k J_k$.

- Reversal charge Q is defined to be the one that makes $\mathcal{I} = 0$.

- Consider simplest $Q(x)$:

$$Q(x) = Q^* \text{ for } x \in [x_1, x_2] \text{ and } Q(x) = 0 \text{ for } x \notin [x_1, x_2].$$

For the given form of $Q(x)$, define

$$g(V) := \sum_{s=1}^{n} \frac{z_s(L_s e^{z_s \mathcal{V}_0} - R_s)}{1 - x_2 + x_1 e^{z_s \mathcal{V}_0} + (x_2 - x_1)e^{z_s V}}.$$

$\mathrm{Theorem}$. For any real root $\mathcal{V}^*$ of $g(V) = 0$, if

$$Q^* = -\sum_{s=1}^{n} z_s e^{z_s(\mathcal{V}_0 - \mathcal{V}^*)} \frac{(1 - x_2 + (x_2 - x_1)e^{z_s \mathcal{V}^*})L_s + x_1 R_s}{1 - x_2 + x_1 e^{z_s \mathcal{V}_0} + (x_2 - x_1)e^{z_s \mathcal{V}^*}},$$

then $\mathcal{I} = 0$ and, for $k = 1, 2, \ldots, n$,

$$J_k = \frac{L_k e^{z_k \mathcal{V}_0} - R_k}{1 - x_2 + x_1 e^{z_k \mathcal{V}_0} + (x_2 - x_1)e^{z_k \mathcal{V}^*}}.$$

Theorem. For $n = 2$, $\exists$ a reversal charge Q^* if and only if

$$(L_1 e^{z_1 \mathcal{V}_0} - R_1)(L_2 e^{z_2 \mathcal{V}_0} - R_2) > 0.$$

The reversal charge Q^* is unique. Moreover,

if $L_1 e^{z_1 \mathcal{V}_0} - R_1 > 0$, then Q^* and $\mathcal{V}_0$ have the same sign;

if $L_1 e^{z_1 \mathcal{V}_0} - R_1 < 0$, then Q^* and $\mathcal{V}_0$ have opposite signs.

Theorem. Let $n = 2$ w/ $z_1 = 1 = -z_2$ ($L_j = L$ and $R_j = R$).

Assume $Le^{-\mathcal{V}_0} - R > 0$ and $\mathcal{V}_0 > 0$ (so that $J_1(0) > J_2(0) > 0$).

For some choices of $(\mathcal{V}_0, L, R)$, one has

$$J_1(0) > J_2(0) > J_1(Q^*) = J_2(Q^*).$$

[Somewhat counterintuitive, if not, nobody knows before.]

Theorem. Let $n = 3$ w/ $z_1 = 1$, $z_2 = 2$ and $z_3 = -1$.

For example, for mixing of $\mathrm{Na^+Cl^-}$ and $\mathrm{Ca^{++}Cl_2^-}$.

For some bdry conditions, there are at least TWO reversal charges.

Theorem. [Reversal Potential]

For any three-piece $Q(x)$, there is at least one reversal potential.

The number of reversal potentials is odd.

2.2. Effects of small $Q(x)$ and channel geometry

Ji-L.-Zhang (Preprint) [From governing system in Eisenberg-L. 07 SIMA]

Consider three-piece $Q(x) = Q_0$ over $[x_1, x_2]$ w/ $n = 2$.

Electroneutrality: $z_1 L_1 = -z_2 L_2 = L$ and $z_1 R_1 = -z_2 R_2 = R$.

$$J_k(Q_0, \varepsilon) = J_{k0} + J_{k1} Q_0 + O(\varepsilon, Q_0^2).$$

2.2.1. Effects of channel geometry on fluxes of zeroth order in Q_0

$$J_{10} = \frac{L - R}{z_1 H(1)(\ln L - \ln R)} \mu_1^\delta, \quad J_{20} = \frac{R - L}{z_2 H(1)(\ln L - \ln R)} \mu_2^\delta;$$

$$\mu_k^\delta := \mu_k(0) - \mu_k(1) = z_k \mathcal{V}_0 + \ln L - \ln R \ \text{ and } \ H(1) = \int_0^1 A^{-1}(x)dx.$$

J_{10} doesn't depend on 2nd ion species and J_{20} doesn't depend on 1st ion species.

Effects of channel geometry w/o Q_0 are simple.

2.2.2. Effects of $Q(x)$ and channel geometry on 1st order fluxes

$$J_{11} = \frac{A(\mu_2^\delta - z_2 BV)}{(z_1 - z_2)H(1)(\ln L - \ln R)^2}\mu_1^\delta,$$
$$J_{21} = \frac{A(\mu_1^\delta - z_1 BV)}{(z_2 - z_1)H(1)(\ln L - \ln R)^2}\mu_2^\delta,$$

where, in terms of $\alpha = H(x_1)/H(1)$ and $\beta = H(x_2)/H(1)$,

$$A = A(L,R) = -\frac{(\beta-\alpha)(L-R)^2}{((1-\alpha)L+\alpha R)((1-\beta)L+\beta R)(\ln L - \ln R)},$$
$$B = B(L,R) = \frac{\ln((1-\beta)L+\beta R) - \ln((1-\alpha)L+\alpha R)}{A}.$$

J_{11} depends on 2nd ion species and J_{21} depends on 1st ion species.

More detailed channel geometry presents in J_{k1}.

2.2.3. Charge and channel geometry effects on fluxes

For $t > 0$, set

$$\gamma(t) = \frac{t \ln t - t + 1}{(t-1)\ln t}.$$

Lemma 1. *For* $t > 0$, $0 < \gamma(t) < 1$, $\gamma'(t) > 0$,

$$\lim_{t\to 0} \gamma(t) = 0, \quad \lim_{t\to 1} \gamma(t) = 1/2, \quad \lim_{t\to\infty} \gamma(t) = 1.$$

Theorem 5. *Let V_q^1 and V_q^2 be as*

$$V_q^1 = V_q^1(L,R) = -\frac{\ln L - \ln R}{z_2(1-B)} \text{ and } V_q^2 = V_q^2(L,R) = -\frac{\ln L - \ln R}{z_1(1-B)}.$$

Then, for $t = L/R > 1$, one has $A < 0$, and

(i) if $\alpha < \gamma(t)$ and $\beta \in (\alpha, \beta_1)$, then $V_q^1 < 0 < V_q^2$; and,

- *(i1) for $V \in (V_q^1, V_q^2)$, positive Q_0* reduces *both $|J_1|$ and $|J_2|$;*
- *(i2) for $V < V_q^1$, positive Q_0* strengthens *$|J_1|$ but* reduces *$|J_2|$;*
- *(i3) for $V > V_q^2$, positive Q_0* reduces *$|J_1|$ but* strengthens *$|J_2|$;*

(ii) if either $\alpha < \gamma(t)$ and $\beta \in (\beta_1, 1)$ or $\alpha \geq \gamma(t)$, then $V_q^1 > 0 > V_q^2$; and,

- *(ii1) for $V \in (V_q^2, V_q^1)$, positive Q_0* strengthens *both $|J_1|$ and $|J_2|$;*
- *(ii2) for $V > V_q^1$, positive Q_0* strengthens *$|J_1|$ but* reduces *$|J_2|$;*
- *(ii3) for $V < V_q^2$, positive Q_0* reduces *$|J_1|$ but* strengthens *$|J_2|$.*

2.2.2. Major effect of channel geometry (through permanent charges)

Theorem 6. *Under some conditions, $|J_{11}|$ and $|J_{21}|$ attain their maximums for $(\alpha, \beta) = (0, 1)$.*

(i). It implies: A short and narrow neck " = " A long and wide neck.

Short and Narrow : $x_2 - x_1 \ll 1$ and $A(x)$ is much smaller for $x \in (x_1, x_2)$.
Long and Wide: $x_2 - x_1 \approx 1$ and $A(x)$ is more uniform.

"Evolution knows and chooses short and narrow necks".

(ii). Without permanent charges, the channel geometry needs not to be complex – only its "average" property $H(1)$ come into the picture for fluxes and current. (True for PNP with HS.)

Thank You !