Efficiency and Stability in Large Matching Markets

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Introduction

An important class of resource allocation problems involves "matching without transfers"

- assignment of students to public school
- allocation of social housing
- assignment of teachers to schools
- assignment of organs to patients in need

In practice, those markets are often organized in a centralized way.

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2. **Stability:** respecting agents' priorities (aka "no justified envy", or "fairness").
   - Attained by Gale and Shapley's Deferred Acceptance Algorithm (DA).

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- DA is stable and efficient among stable mechanisms (Gale and Shapley, 62)
  (Boston, Hong Kong, New York, Paris...)
- Top Trading Cycle is efficient and envy minimal (Abdulkadiroglu, Che, Tercieux, 13)
  (San Francisco, New Orleans,...)

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- What is the optimal way to resolve the tradeoff of the two goals? Attaining one at the minimal sacrifice of the other may not be the best if the sacrifice is significant and/or if one can approximately achieve both.

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- Large markets: Realistic in the applications mentioned. In New York, 100,000 students apply each year to 500 schools; In medical matching, 20,000 doctors and 3,000-4,000 programs
- Random preference structure: individuals draw preferences at random with some correlation (to be specified).

Setting

- Finite set of individuals I and finite set of objects O to be matched
  - For simplicity, $|I| = |O| = n$

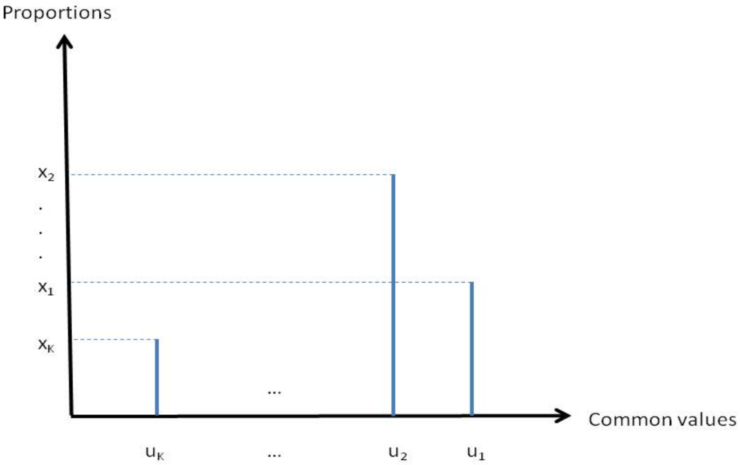
Setting: preferences

Each $i \in I$ receives utility from object $o \in O$

$$U_i(o) = U(u_o, \xi_{io})$$

- where u_o is the *common value component*
- The u_o are in $[0, 1]$
- Let $X^n(\cdot)$ be its distribution and $X(\cdot)$ its limit

Distribution of common values (finite example)



Setting: preferences

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- ξ_{io} is the *idiosyncratic shock* on i's preferences for object o
- The $\{\xi_{io}\}_{i,o}$ is a collection of iid random variable
  Distribution takes values in $[0, \bar{\xi}] \subset \mathbb{R}$
- $U(\cdot, \cdot)$ takes values in $\mathbb{R}_+$, is strictly increasing and continuous
- All objects are acceptable
  (utility of the outside option is normalized to 0)

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Purely idiosyncratic preferences.

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- A **matching mechanism** $\tilde{\mu}$ maps "states" into matchings, where a state refers to the profile of preferences together with the profile of priorities.

PE mechanism: Serial Dictatorship (SD) mechanism

A serial dictatorship mechanism SD^f specifies an ordering $f : \{1, 2, 3, ..., n\} \to I$, where $f(i)$ is the i^{th} "dictator"

- $f(1)$ chooses his favorite object
- $f(2)$ chooses his favorite object among the remaining ones
- and so on....

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Step $t = 2, \ldots$**:** Repeat the same procedure with the remaining economy.

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Theorem

Let μ be a Pareto-efficient matching mechanism.

$$\frac{1}{n} \sum_{i \in I} U_i(\mu(i)) \xrightarrow{p} U^*,$$

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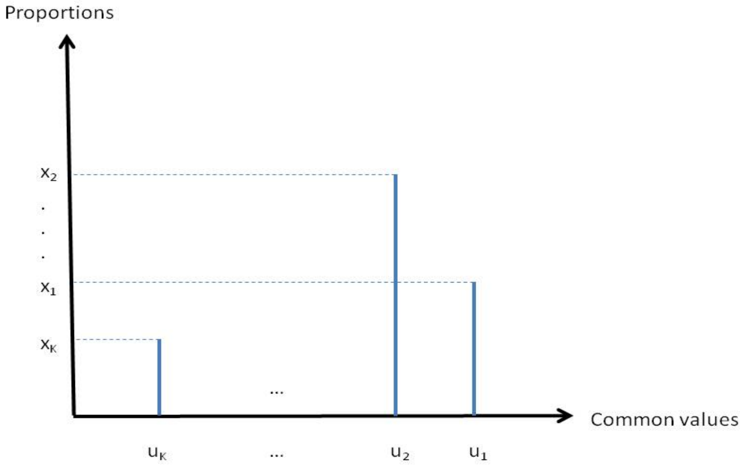
Let μ be a Pareto-efficient matching mechanism.

$$\frac{1}{n} \sum_{i \in I} U_i(\mu(i)) \xrightarrow{p} U^*,$$

i.e., for any $\delta > 0$,

$$\Pr \left\{ \left| \frac{1}{n} \sum_{i \in I} U_i(\mu(i)) - U^* \right| < \delta \right\} \to 1 \text{ as } n \to \infty.$$

Implication in terms of distribution of payoffs:



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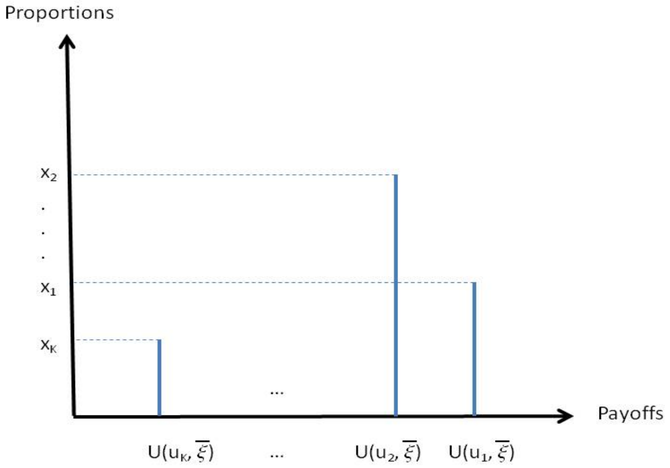

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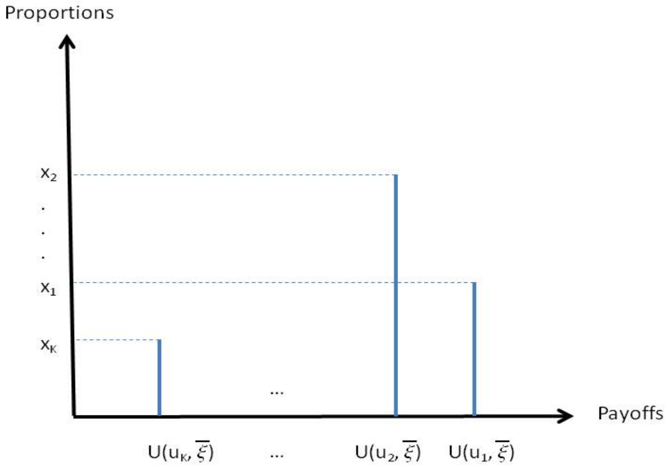

Sketch of proof

Intuition given for the case where $X(\cdot)$ is degenerate (i.e. we only have the idiosyncratic component)

A PE mechanism $\tilde{\mu}$ can be implemented by a serial dictatorship mechanism with a particular serial order $\tilde{f}$

For arbitrarily small $\varepsilon, \delta > 0$, define the random set:

$$\bar{I} := \{ i \in I \,\big|\, U_i(\tilde{\mu}(i)) \leq U(u^0, \bar{\xi}) - \varepsilon \text{ and } \tilde{f}(i) \leq (1-\delta)|O| \}.$$

We show via applying a random graph theory result that

$$\frac{|\bar{I}|}{n} \xrightarrow{p} 0.$$

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(We show $\bar{I}$ to be a shorter side of an independent set of an associated random graph, which vanishes.)

Detour: Random bipartite graph

A random bipartite graph $G(V_1, V_2, p)$:

- V_1 is the set of vertices on one side
- V_2 is the set of vertices on the other side and

The set of edges is random:

- An edge $(i, j) \in V_1 \times V_2$ is added with probability p.

Size of an independent set

Given a (deterministic) bipartite graph $G(V_1, V_2, E)$,

- $W_1 \times W_2 \subseteq V_1 \times V_2$ is an independent set if

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Theorem (Extension of Bollobas and Erdös (1975))

Let $W_1 \times W_2$ be an independent set in a random bipartite graph $G(V_1, V_2, p)$ where $0 < p < 1$

$$\Pr \{\min \{|W_1|, |W_2|\} < \kappa \ln n\} \to 1 \text{ as } n \to \infty.$$

(where κ is a strictly positive constant)

Sketch of proof

Now that we have

$$\bar{I} := \{ i \in I \,\big|\, U_i(\tilde{\mu}(i)) \leq \bar{\bar{\xi}} - \varepsilon \text{ and } \tilde{f}(i) \leq (1-\delta)|O| \}$$

let us define

$$\bar{O} := \{ o \in O \,\big|\, \tilde{f}(\tilde{\mu}(o)) \geq (1-\delta)|O| \}.$$

Build an associated random bipartite graph

Random variables $\{\xi_{io}\}$ induce a random graph on $I \times O$ where

$$(i, o) \text{ is an edge iff } \xi_{io} > \bar{\bar{\xi}} - \varepsilon$$

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objects assigned to agents with "bad" serial orders.

Sketch of proof

Claim. $\bar{I} \times \bar{O}$ is an independent set in the associated random graph.

Proof. Otherwise, if $(i, o) \in \bar{I} \times \bar{O}$ is an edge then

1. $(i, o) \in \bar{I} \times \bar{O} \Longrightarrow U_i(o) > U_i(\tilde{\mu}(i))$
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$\Longrightarrow$ i picks $\tilde{\mu}(i)$ while a better object o is available. Contradiction.

Stability versus efficiency

Asymptotic Efficiency and Stability

Matching mechanism $\tilde{\mu}$ is **asymptotically efficient** if

for any $\tilde{\mu}'$ which Pareto-dominates $\tilde{\mu}$ and any $\epsilon > 0$

$$\frac{|I_\epsilon(\tilde{\mu}'|\tilde{\mu})|}{|I|} \xrightarrow{p} 0,$$

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Matching mechanism $\tilde{\mu}$ is **asymptotically stable** if, for any $\epsilon > 0$

$$\frac{|J_\epsilon|}{|I \times O|} \xrightarrow{p} 0,$$

where

$$J_\epsilon := \{(i,o) \in I \times O | U_i(o) - U_i(\tilde{\mu}(i)) > \epsilon \text{ and } V_o(i) - V_o(\tilde{\mu}(o)) > \epsilon\}.$$

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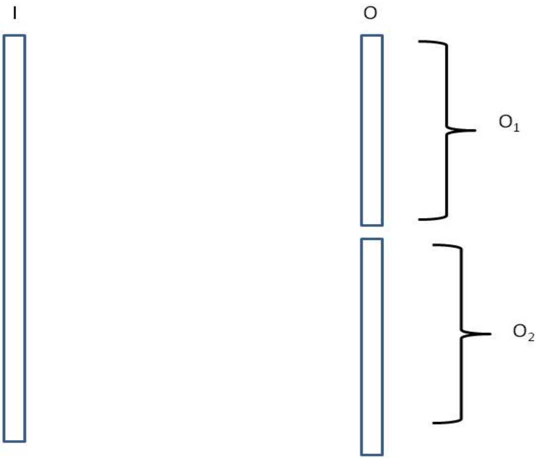
- If $X(\cdot)$ is degenerate (i.e., only one tier of objects), TTC is asymptotically stable: Our first result implies that all individuals get a payoff arbitrarily close to the upper bound $U(u_1^0, \bar{\xi})$
- But if we add tiers on objects/correlation in individuals' preferences, TTC is not asymptotically stable (even with this weaker notion).

Asymptotic Instability of TTC



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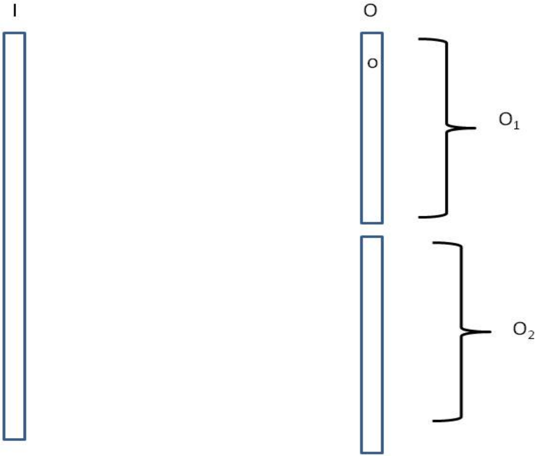

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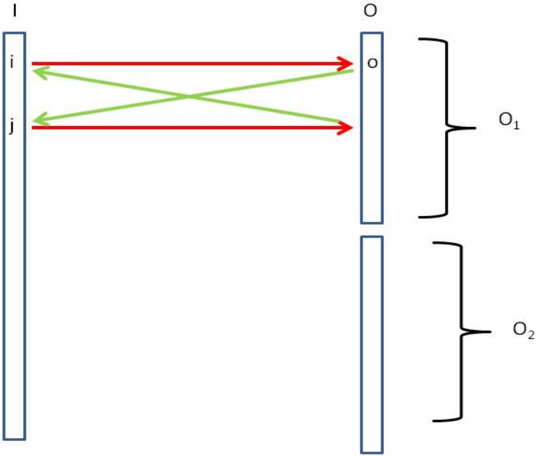

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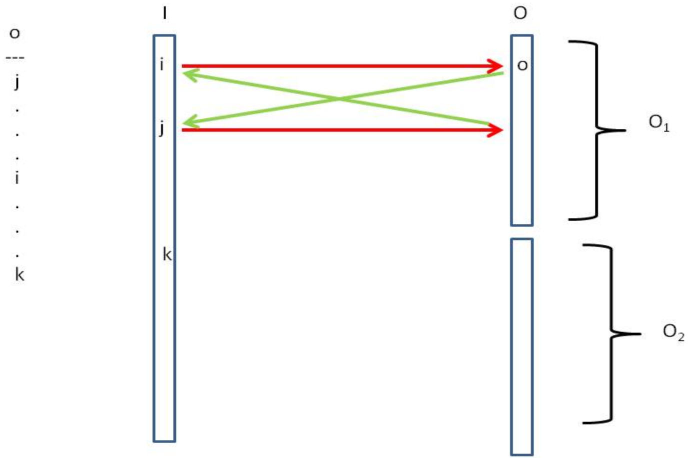

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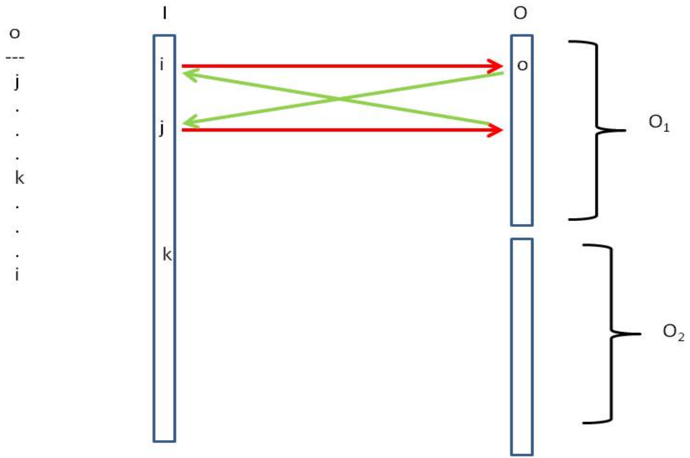

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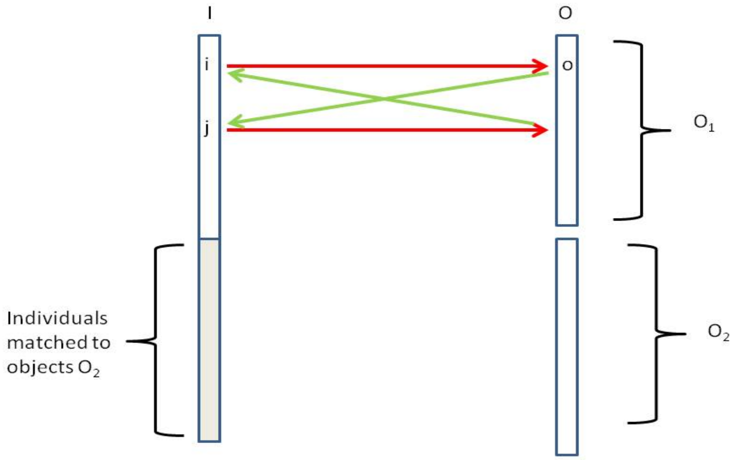

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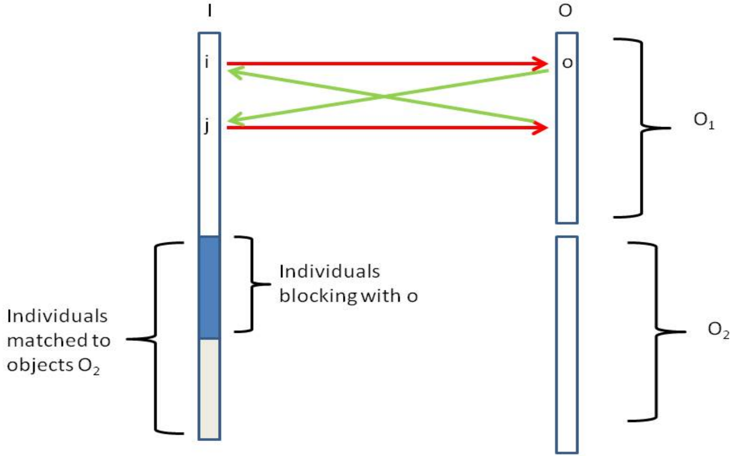

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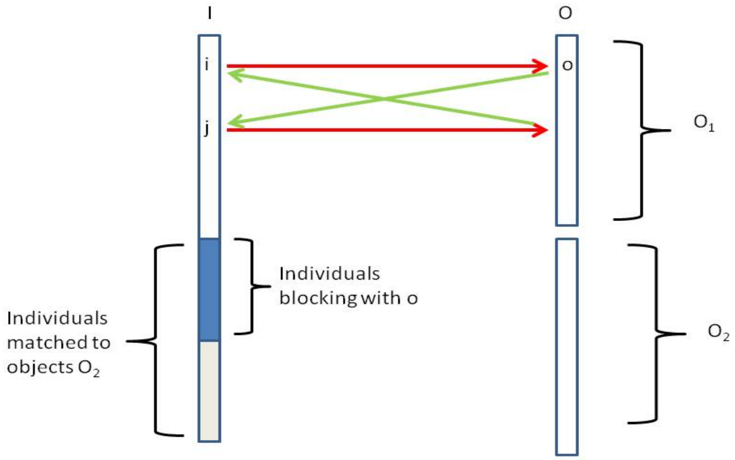

$\Longrightarrow$ nb. of blocking pairs $\geq n/2$ $(n/4)$

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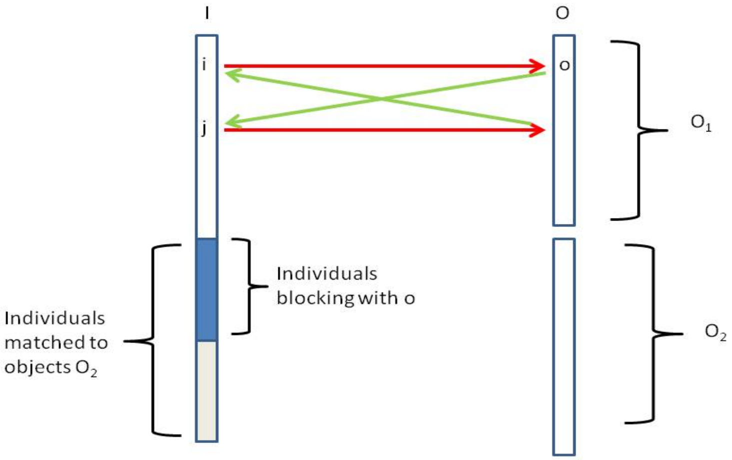

$\Longrightarrow$ More than 12.5% of blocking pairs even for very large markets

Asymptotic Inefficiency of DA

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  - Apply Ashlagi, Kanoria, and Leshno (2013).

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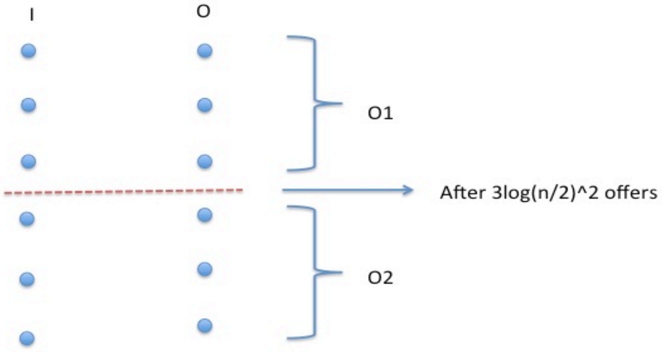

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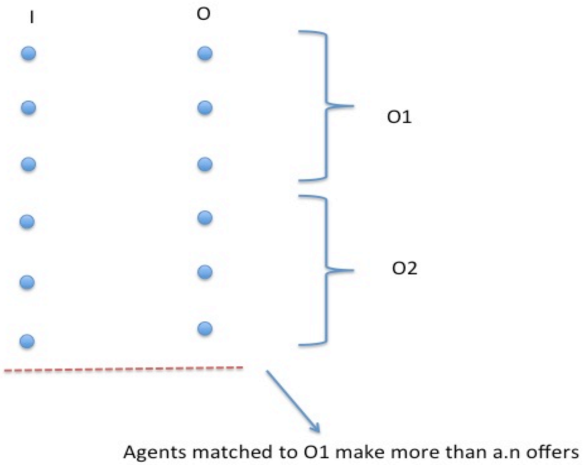

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  - Keep track of the number of offers made by each individual.
  - When there is an individual who has made more than $\beta(n)$ offers, finalize the matching, i.e., any object gets matched with the individual he tentatively holds if any.

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  - Start running the McVitie-Wilson version of Gale-Shapley's algorithm.
  - Keep track of the number of offers made by each individual.
  - When there is an individual who has made more than $\beta(n)$ offers, finalize the matching, i.e., any object gets matched with the individual he tentatively holds if any.
- Iterate the process until we exhaust the market.

Result

Theorem

Let $\tilde{\mu}$ be the matching mechanism obtained by this procedure for $\lceil \log(n)^2 \rceil \leq \beta(n) = o(n)$. $\tilde{\mu}$ is asymptotically efficient and asymptotically stable.

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The mechanism is "asymptotically incentive compatible": Truthtelling is an ϵ-Bayes Nash equilibrium.

Intuition for the result

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3. **whp**, individuals matched in this step get high idiosyncratic payoffs
4. **whp**, almost all objects in O_1 get high idiosyncratic payoffs (by the classical results).

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Let $\tilde{\mu}$ be the matching mechanism obtained by this procedure for $\lceil \log(n)^2 \rceil \leq \beta(n) = o(n)$. $\tilde{\mu}$ is asymptotically efficient and asymptotically stable.

We iterate the reasoning for other tiers.

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Theorem

Let $\tilde{\mu}$ be the matching mechanism obtained by this procedure for $\lceil \log(n)^2 \rceil \leq \beta(n) = o(n)$. $\tilde{\mu}$ is asymptotically efficient and asymptotically stable.

Hence,

1. **whp**, almost all objects get high payoffs $\Longrightarrow$ asymptotically stable
2. **whp**, all individuals are assigned objects that yield high idiosyncratic payoffs $\Longrightarrow$ asymptotically efficient

Simulations

We simulate a situation where

- common values uniformly distributed from $[0, 1]$
- the idiosyncratic payoff ξ_{io} uniformly distributed from $[0, 1]$

Conclusion

- While there is an (asymptotically) efficient and stable matching mechanism: two of the prominent mechanisms fail to find this matching
- Alternative mechanism which limits competition seem to perform better
- In practice, students can only report a small number of objects in their list of preferences. This also limits the total number of offers that agents can make, and this may an unexpected good effect on the performance of the mechanism

Conclusion: Data ongoing

Abdulkadiroglu, Pathak and Roth (2006) have studied NYC data for the entrance in high school

- Under DA: out of 80,000 students 5,000 can be made better-off by letting them exchange their assignments
- Under TTC: out of 80,000, 55,000 are part of a blocking pair

This suggests that DA and TTC are indeed not close to be efficient or stable in the field.

We are currently running our alternative algorithm on NYC data...