The Network Expansion Problem with Non-linear Costs

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- Designing the new links' capacities without improving the existing link facilities,
- Minimizing the summation of two costs, the performance costs of existing and new links and the construction costs of the new links,
- The network expansion problem is to find the minimum of the difference of convex functions over the linear constraints.
- Tuy(1987) proposed a method for the D.C. problem : the problem is transformed to a concave minimization over a convex feasible set.

Introduction

- Network models,
  - *The minimum cost routing problem*, where the objective is to find the optimum way of routing traffic through a given network, satisfying given demands,
  - *The network design problem*, where the network and the capacities of the links should be planned according to the flow pattern.
- The costs,
  - constructing cost of the linking facilities (known as *construction or design cost*),
  - the routing cost of the network (known as *performance cost*).
- Network design problems:
  - new network design
  - network expansion
  - network improvement

The Network Expansion Problem

$$Z = \min \quad \sum_{l\in L_1} x_l D_l(\tfrac{x_l}{k_l}) + \sum_{l\in L_2}(y_l D_l(\tfrac{y_l}{c_l}) + C_l(c_l)) \tag{1}$$

$$s.t. \quad \begin{cases} \sum_{l\in S_i} f_l^k - \sum_{l\in E_i} f_l^k = d_i^k, \forall k \in K, i \in N \\ \sum_{k\in K} f_l^k = \begin{cases} x_l & ,\forall l \in L_1 \\ y_l & ,\forall l \in L_2 \end{cases} \\ f_l^k \geq 0 & ,\forall k \in K, l \in L_1 \cup L_2 \\ c_l \geq 0 & ,\forall l \in L_2 \end{cases} \tag{2}$$

in which

N : the set of nodes,
K: $the set of source nodes$,
M: $the set of destination nodes$,
d_i^k : the traffic demand on i from k,
f_l^k : the flow on link l originated from k,
x_l : the total flow on the existing link l,
y_l : the total flow on the new link l,
D_l : the performance cost function for link l,
C_l : the construction cost function for the new link l.

L_1 : the set of existing links,
L_2 : the set of new links,
S_i : the set of all link with the start node i,
E_i : the set of all links with the end node i,

k_l : the existing capacity for the link l,
c_l : the required capacity for the new link l,

- Decomposing the problem on two variable sets y and c and defining

$$H_l(x_l) = x_l D_l(\frac{x_l}{k_l}) \quad ,\forall l \in l_1,$$

$$H_l(y_l) = \min_{c_l \geq 0} y_l D_l(\frac{y_l}{c_l}) + C_l(c_l) \quad ,\forall l \in l_2$$

- we have:

$$Z = \min \sum_{l \in L_1} H_l(x_l) - \sum_{l \in L_2}(-H_l(y_l)) \tag{3}$$

$$s.t. \begin{cases} \sum_{l \in S_i} f_l^k - \sum_{l \in E_i} f_l^k = d_i^k, \forall k \in K, i \in N \\ \sum_{k \in K} f_l^k = \begin{cases} x_l & ,\forall l \in L_1 \\ y_l & ,\forall l \in L_2 \end{cases} \\ f_l^k \geq 0 & ,\forall k \in K, l \in L_1 \cup L_2 \end{cases} \tag{4}$$

- H_l is convex for $l \in L_1$ and is concave for $l \in L_2$ and the network expansion problem will be equivalent to a flow problem with the objection function as a difference of two convex functions.

Tuy Method for D.C. problem with linear constraints

- Consider the following d.c. optimization problem:

$$(P) \qquad \min \qquad f(x) - g(y)$$
$$s.t. \quad \begin{cases} Ax + By + c = 0 \\ x \in X, y \in Y \end{cases}$$

- Introducing the supplementary variable t :

$$(P) \qquad \min \qquad t - g(y)$$
$$s.t. \quad \begin{cases} f(x) \leq t \\ Ax + By + c = 0 \\ x \in X, y \in Y \end{cases}$$

- equivalent to:

$$(Q) \qquad \min \quad t - g(y)$$
$$(y, t) \in D$$

- in which

$$Y_0 = \{y \in Y : \exists x \in X Ax + By + c = 0\}$$
$$\varphi(y) = \inf\{f(x) : Ax + By + c = 0, x \in X\}$$
$$D = \{(y, t) : \varphi(y) \leq t, y \in Y_0\}$$

and D is convex.

Algorithm

- **step 0**

  Select the polyhedron S_0 s.t. : $Y_0 \subseteq S_0 \subseteq Y$
  and choose an arbitrary $y^0 \in Y_0$.

  Solve the convex problem $c(y^0)$ as:

$$(c(y^k)) \qquad \min \quad f(x) \qquad s.t. \begin{cases} Ax + By^k + c = 0 \\ x \in X \end{cases}$$

  If $\varphi(y^0) = -\infty$, then problem (Q) is infinite (Stop).
  Otherwise, let λ^0 be the kuhn-tucker multipliers vector and :

$$T_1 = \{(y,t) : y \in S_0, \lambda^o B(y - y^0) + \varphi(y^0) - t \leq 0\}$$

  and $k \leftarrow 1$.

- **step 1**

  Solve the following relaxed problem. Let its optimal solution be (y^k, t^k).

$$(Q_k) \qquad \min \quad t - g(y) \qquad (y,t) \in T_k$$

- **step 2**

  Solve the following linear program $(R^*(y^k))$ (solution: μ^k and γ^k).

$$(R^*(y^k)) \quad \max \quad \mu(By^k\mu + c) + \gamma d$$
$$s.t. \begin{cases} -\mu A + \gamma E = 0 \\ \mu e \leq 1 \\ \gamma \geq 0 \end{cases}$$

  It is the dual to the problem $(R(y^k))$ in which the optimal solution $\theta = 0$ would be equivalent to the feasibility condition $y^k \in Y_0$:

$$(R(y^k)) \quad \min \quad \theta$$
$$s.t. \begin{cases} Ax + By^k + c = e\theta \\ x \in X, \theta \geq 0 \end{cases}$$

  If $\mu^k = \gamma^k = 0$, then go to step 3, otherwise go to step 4.

- **step 3**

  Solve the convex problem $(c(y^k))$ (solution: $y^k, \varphi(y^k)$).
  If $\varphi(y^k) = -\infty$, then problem (Q) is infinite (Stop).
  If $\varphi(y^k) \leq t^k$, then (y^k, t^k) is optimal to (Q) and (p) (Stop).
  Otherwise, $\varphi(y^k) > t^k$, then let λ^k be the kuhn-tucker multipliers for $c(Y^k)$ and add the following constraint to T_k.

$$\lambda^k B(y - y^k) + \varphi(y^k) - t \leq 0$$

  Let $k \leftarrow k + 1$ and return to step 1.

- **step 4**

  Add the following constraint to T_k.

  $$\mu^k(By + c) + \gamma^k d \leq 0$$

  Let $k \leftarrow k + 1$ and return to step 1.

$$Z = \min \quad \sum_{l\in L_1} H_l(x_l) - \sum_{l\in L_2}(-H_l(y_l)) \tag{5}$$

$$s.t. \quad \begin{cases} \sum_{l\in S_i\cap L_1} x_l + \sum_{l\in S_i\cap L_2} y_l - \sum_{l\in E_i\cap L_1} x_l - \sum_{l\in E_i\cap L_2} y_l = d_i, \forall i \in N \\ x_l \geq 0 \quad, \forall l \in L_1 \qquad , y_l \geq 0 \quad , \forall l \in L_2 \end{cases} \tag{6}$$

Consider the convex problem as follows:

$$(c(y^k)) \min \quad \sum_{l\in L_1} H_l(x_l) \tag{7}$$

$$s.t. \quad \begin{cases} \sum_{l\in S_i\cap L_1} x_l - \sum_{l\in E_i\cap L_1} x_l = d_i^{'}, \forall i \in N & (u_i) \\ x_l \geq 0 \quad , \forall l \in L_1 & (v_l) \end{cases} \tag{8}$$

If x^* is the optimal solution to problem $(c(y^k))$, then its lagrangian multipliers, u_i and v_l is the solution of the following system of equations:

$$\begin{array}{ll} H_l^{'}(x_l^*) + u_{i_l} - U_{j_l} - v_l = 0 & \text{for} \quad l \in L_1 \\ x_l^* v_l = 0 & \text{for} \quad l \in L_1 \\ v_l \geq 0 & \text{for} \quad l \in L_1 \end{array} \tag{9}$$

it is not necessary to solve problem $(R^*(y^k))$,because for each $y^k \in Y$, there is one $x \in X$ such that the flow conservation constraint holds, and therefore there exists one $y^k \in Y_0$.

Numerical Example

Consider a network of 6 nodes and 16 existing links and 2 new links. Assume that the supply at node 1 is 10 and the demand at node 6 is 10.

$D_l(\frac{x_l}{k_l}) = a_l + b_l(\frac{x_l}{k_l})^4$

$x_l(c_l) = g_l\sqrt{c_l}$

$$H_l(x_l) = x_l(a_l + b_l(\frac{x_l}{k_l})^4) \quad , l \in l_1$$

and

$$H_l(y_l) = y_l(a_l + b_l(\frac{y_l}{(8b_l y_l^5/g_l)^(2/9)})^4 + g_l((8b_l y_l^5/g_l)^{(1/9)}) \quad , l \in l_2.$$

Data

Link no.	start node	end node	a_l	b_l	k_l
1	1	2	1	10	3
2	1	3	2	5	10
3	2	1	3	3	9
4	2	3	4	20	4
5	2	4	5	50	3
6	3	1	2	20	2
7	3	2	1	10	1
8	3	5	1	1	10
9	4	2	2	8	45
10	4	5	3	3	3
11	4	6	9	2	2
12	5	3	4	10	6
13	5	4	4	25	44
14	5	6	2	33	20
15	6	4	5	5	1
16	6	5	6	1	4.5

The Data for the existing links in the test example

Link no.	start node	end node	a_l	b_l	g_l
1	2	5	0.4	0.5	0.2
2	3	4	0.5	0.5	0.25

The Data for the new links in the test example intermediate result

- $Y_0 = \{(y_1, y_2)|0 \leq y_1 \leq 10, 0 \leq y_2 \leq 10\}$

- initial feasible flow $y^0 = (0, 0)$

- The flow on the existing links is determined by Frank & Wolfe method as $x^0 = (1.90, 8.10, 0, 0.79, 1.11, 0, 0, 8.89, 0, 0, 1.11, 0, 0, 8.89, 0, 0)$ with the objective value of $\varphi(y^0) = 102.25$.

- The corresponding lagrangian multipliers is : $\lambda^0 = (0, 8.59, 12.75, 15.35, 16.87, 25.31)$.

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  $T_1 = \{(y, t)|0 \leq y_1 \leq 10, 0 \leq y_2 \leq 10, 0 \leq t \leq 100000, 8.27y_1 + 2.61y_2 + t \geq 102.25\}$

  then, the solution to the concave problem with the feasible set T_1 would be $y_1^1 = 10.01$, $Y_2^1 = 7.45$ and $t^1 = 0.00$ with the objective value of $t^1 - g(y^1) = 10.03$.

- $\varphi(y^1) > t^1 + 0.005\varphi(y^1)$, a cut is added to separated y^1,the flow on the new links , from T_1.

- The cuts which have been added for the solutions of the concave problem are as follows:

$$\begin{aligned}
-6.03y_1 + 0.49y_2 + t &\geq 37.73\\
17.92y_1 - 19.04y_2 + t &\geq 16.25\\
4.21y_1 - 5.19y_2 + t &\geq 84.33\\
-6.00y_1 + 6.21y_2 + t &\geq 72.60\\
0.71y_1 + 0.98y_2 + t &\geq 88.56\\
-1.60y_1 + 0.69y_2 + t &\geq 83.20\\
1.18y_1 - 1.06y_2 + t &\geq 87.51\\
0.93y_1 - 1.11y_2 + t &\geq 90.65
\end{aligned}$$

Result

k	y_1^k	y_2^k	t^k	$t^k - g(y^k)$	$\varphi(y^k)$	$\varphi(y^k) - g(y^k)$	$\varphi(y^k) - t^k$	UBD
1	10.01	7.45	0.00	10.03	94.56	104.59	94.56	102.25
2	3.03	10.00	51.12	59.28	152.52	160.68	101.40	102.25
3	3.68	5.61	57.18	63.10	98.05	103.97	40.87	102.25
4	4.55	0.00	65.16	67.71	99.99	102.54	34.83	102.25
5	2.35	1.08	80.01	82.39	85.91	88.29	5.90	88.29
6	5.22	3.64	81.25	86.73	89.09	94.57	7.84	88.29
7	2.08	1.87	85.23	88.01	87.13	89.91	1.90	88.29
8	2.19	1.01	86.00	88.25	89.81	92.06	3.81	88.29
9	2.60	0.00	88.22	89.79	89.31	90.88	1.09	88.29

The result for each iteration of Tuy method for the test example

Optimal solution

$$x^* = (2.09, 7.91, 0, 0, 0, 0, 0.26, 6.57, 0, 0.06, 1.02, 0, 0, 8.98, 0, 0)$$

$$y^* = (2.35, 1.08)$$

The designed capacities for the new links are determined by definition from H and $c^* = (5.03, 2.02)$.