Bi-Lipschitz Solutions to the Prescribed Jacobian Inequality in the Plane and Applications to Nonlinear Elasticity

Olivier Kneuss

joint work with Julian Fischer (MPI Leibzig)

Fields Institute
Toronto

30.9.2014

The Prescribed Jacobian Equation

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $f : \overline{\Omega} \to \mathbb{R}$, $n \geq 2$.

Can we find a map $\phi : \overline{\Omega} \to \mathbb{R}^n$ satisfying

$$\begin{cases} \det \nabla \phi = f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega? \end{cases} \tag{1}$$

Obvious necessary condition:

$$\int_\Omega f = |\Omega|.$$

The Prescribed Jacobian Equation

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $f : \overline{\Omega} \to \mathbb{R}$, $n \geq 2$.
Can we find a map $\phi : \overline{\Omega} \to \mathbb{R}^n$ satisfying

$$\begin{cases} \det \nabla\phi = f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega? \end{cases} \tag{1}$$

Obvious necessary condition:

$$\int_\Omega f = |\Omega|.$$

The Prescribed Jacobian Equation

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $f : \overline{\Omega} \to \mathbb{R}$, $n \geq 2$.
Can we find a map $\phi : \overline{\Omega} \to \mathbb{R}^n$ satisfying

$$\begin{cases} \det \nabla\phi = f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega? \end{cases} \tag{1}$$

Obvious necessary condition:

$$\int_\Omega f = |\Omega|.$$

The Prescribed Jacobian Equation: Exsitence Theory

Existence results when f is regular enough (Hölder continuous):

- $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$, $r \geq 0$, $0 < \alpha < 1$

$\Rightarrow$ Existence of $\phi \in C^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ satisfying (1): Dacorogna-Moser '90, also Rivière-Ye '96 and Carlier-Dacorogna '13.

- $f \in W^{m,p}(\Omega)$, $\inf f > 0$, with $m \geq 1$ and $p > \max\{1, n/m\}$

$\Rightarrow$ Existence of $\phi \in W^{m+1,p}(\Omega;\Omega)$ satisfying (1): Ye '94.

- $f \in C^{r,\alpha}(\overline{\Omega})$, no sign hypothesis on f, $r \geq 1$, $0 \leq \alpha \leq 1$

$\Rightarrow$ Existence of $\phi \in C^{r,\alpha}(\overline{\Omega};\mathbb{R}^n)$ satisfying (1): Cupini-Dacorogna-K '09.

The Prescribed Jacobian Equation: Exsitence Theory

Existence results when f is regular enough (Hölder continuous):

- $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$, $r \geq 0$, $0 < \alpha < 1$

$\Rightarrow$ Existence of $\phi \in C^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ satisfying (1): Dacorogna-Moser '90, also Rivière-Ye '96 and Carlier-Dacorogna '13.

- $f \in W^{m,p}(\Omega)$, $\inf f > 0$, with $m \geq 1$ and $p > \max\{1, n/m\}$

$\Rightarrow$ Existence of $\phi \in W^{m+1,p}(\Omega;\Omega)$ satisfying (1): Ye '94.

- $f \in C^{r,\alpha}(\overline{\Omega})$, no sign hypothesis on f, $r \geq 1$, $0 \leq \alpha \leq 1$

$\Rightarrow$ Existence of $\phi \in C^{r,\alpha}(\overline{\Omega};\mathbb{R}^n)$ satisfying (1): Cupini-Dacorogna-K '09.

The Prescribed Jacobian Equation: Exsitence Theory

Existence results when f is regular enough (Hölder continuous):

- $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$, $r \geq 0$, $0 < \alpha < 1$

$\Rightarrow$ Existence of $\phi \in C^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ satisfying (1): Dacorogna-Moser '90, also Rivière-Ye '96 and Carlier-Dacorogna '13.

- $f \in W^{m,p}(\Omega)$, $\inf f > 0$, with $m \geq 1$ and $p > \max\{1, n/m\}$

$\Rightarrow$ Existence of $\phi \in W^{m+1,p}(\Omega;\Omega)$ satisfying (1): Ye '94.

- $f \in C^{r,\alpha}(\overline{\Omega})$, no sign hypothesis on f, $r \geq 1$, $0 \leq \alpha \leq 1$

$\Rightarrow$ Existence of $\phi \in C^{r,\alpha}(\overline{\Omega};\mathbb{R}^n)$ satisfying (1): Cupini-Dacorogna-K '09.

The Prescribed Jacobian Equation: Exsitence Theory

Existence results when f is regular enough (Hölder continuous):

- $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$, $r \geq 0$, $0 < \alpha < 1$

$\Rightarrow$ Existence of $\phi \in C^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ satisfying (1): Dacorogna-Moser '90, also Rivière-Ye '96 and Carlier-Dacorogna '13.

- $f \in W^{m,p}(\Omega)$, $\inf f > 0$, with $m \geq 1$ and $p > \max\{1, n/m\}$

$\Rightarrow$ Existence of $\phi \in W^{m+1,p}(\Omega;\Omega)$ satisfying (1): Ye '94.

- $f \in C^{r,\alpha}(\overline{\Omega})$, no sign hypothesis on f, $r \geq 1$, $0 \leq \alpha \leq 1$

$\Rightarrow$ Existence of $\phi \in C^{r,\alpha}(\overline{\Omega};\mathbb{R}^n)$ satisfying (1): Cupini-Dacorogna-K '09.

The Prescribed Jacobian Equation: Exsitence Theory

Existence results when f is regular enough (Hölder continuous):

- $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$, $r \geq 0$, $0 < \alpha < 1$

$\Rightarrow$ Existence of $\phi \in C^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ satisfying (1): Dacorogna-Moser '90, also Rivière-Ye '96 and Carlier-Dacorogna '13.

- $f \in W^{m,p}(\Omega)$, $\inf f > 0$, with $m \geq 1$ and $p > \max\{1, n/m\}$

$\Rightarrow$ Existence of $\phi \in W^{m+1,p}(\Omega;\Omega)$ satisfying (1): Ye '94.

- $f \in C^{r,\alpha}(\overline{\Omega})$, no sign hypothesis on f, $r \geq 1$, $0 \leq \alpha \leq 1$

$\Rightarrow$ Existence of $\phi \in C^{r,\alpha}(\overline{\Omega};\mathbb{R}^n)$ satisfying (1): Cupini-Dacorogna-K '09.

The Prescribed Jacobian Equation: Exsitence Theory

Existence results when f is regular enough (Hölder continuous):

- $f \in C^{r,\alpha}(\overline{\Omega})$, $f > 0$, $r \geq 0$, $0 < \alpha < 1$

$\Rightarrow$ Existence of $\phi \in C^{r+1,\alpha}(\overline{\Omega}; \overline{\Omega})$ satisfying (1): Dacorogna-Moser '90, also Rivière-Ye '96 and Carlier-Dacorogna '13.

- $f \in W^{m,p}(\Omega)$, $\inf f > 0$, with $m \geq 1$ and $p > \max\{1, n/m\}$

$\Rightarrow$ Existence of $\phi \in W^{m+1,p}(\Omega; \Omega)$ satisfying (1): Ye '94.

- $f \in C^{r,\alpha}(\overline{\Omega})$, no sign hypothesis on f, $r \geq 1$, $0 \leq \alpha \leq 1$

$\Rightarrow$ Existence of $\phi \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ satisfying (1): Cupini-Dacorogna-K '09.

The Prescribed Jacobian Equation: Exsitence Theory

When f is not regular enough (continuous or less):

- Non existence result (Burago-Kleiner '98, McMullen '98): for all $\varepsilon > 0$ there exists $f \in C^0(\overline{\Omega})$ with $\|f - 1\|_{L^\infty} \leq \varepsilon$ for which there exists no Lipschitz solution to (1).
- Rivière-Ye '96: $f \in C^0(\overline{\Omega})$, $f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega}; \overline{\Omega})$
  $f \in L^\infty(\Omega)$, $\inf f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<\beta} C^{0,\alpha}(\overline{\Omega}; \overline{\Omega})$ for some $\beta \leq 1$ depending on $\|f - 1\|_{L^\infty}$
- Monge-Ampère theory: $f \in C^0 \Rightarrow$, $f > 0$ $\exists\, u \in \cap_{p<\infty} W^{2,p}_{loc}$ with $\det \nabla^2 u = f$ (Caffarelli)
  $f \in L^\infty$, $\inf f > 0 \Rightarrow \exists\, u \in W^{2,1+\varepsilon}_{loc}$ with $\det \nabla^2 u = f$ (De-Phillipis-Figalli).
- Open problem: does there exist a $W^{1,p}$ solution of (1) for some p when f is only C^0 (and positive)?

The Prescribed Jacobian Equation: Exsitence Theory

When f is not regular enough (continuous or less):

- Non existence result (Burago-Kleiner '98, McMullen '98): for all $\varepsilon > 0$ there exists $f \in C^0(\overline{\Omega})$ with $\|f - 1\|_{L^\infty} \leq \varepsilon$ for which there exists no Lipschitz solution to (1).
- Rivière-Ye '96: $f \in C^0(\overline{\Omega})$, $f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega}; \overline{\Omega})$
  $f \in L^\infty(\Omega)$, $\inf f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<\beta} C^{0,\alpha}(\overline{\Omega}; \overline{\Omega})$ for some $\beta \leq 1$ depending on $\|f - 1\|_{L^\infty}$
- Monge-Ampère theory: $f \in C^0 \Rightarrow$, $f > 0$ $\exists\, u \in \cap_{p<\infty} W^{2,p}_{loc}$ with $\det \nabla^2 u = f$ (Caffarelli)
  $f \in L^\infty$, $\inf f > 0 \Rightarrow \exists\, u \in W^{2,1+\varepsilon}_{loc}$ with $\det \nabla^2 u = f$ (De-Phillipis-Figalli).
- Open problem: does there exist a $W^{1,p}$ solution of (1) for some p when f is only C^0 (and positive)?

The Prescribed Jacobian Equation: Exsitence Theory

When f is not regular enough (continuous or less):

- Non existence result (Burago-Kleiner '98, McMullen '98): for all $\varepsilon > 0$ there exists $f \in C^0(\overline{\Omega})$ with $\|f - 1\|_{L^\infty} \leq \varepsilon$ for which there exists no Lipschitz solution to (1).
- Rivière-Ye '96: $f \in C^0(\overline{\Omega})$, $f > 0$, $\Rightarrow \exists\ \phi \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$
  $f \in L^\infty(\Omega)$, $\inf f > 0$, $\Rightarrow \exists\ \phi \in \cap_{\alpha<\beta} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$ for some $\beta \leq 1$ depending on $\|f - 1\|_{L^\infty}$
- Monge-Ampère theory: $f \in C^0 \Rightarrow$, $f > 0\ \exists\ u \in \cap_{p<\infty} W^{2,p}_{loc}$ with $\det \nabla^2 u = f$ (Caffarelli)
  $f \in L^\infty$, $\inf f > 0 \Rightarrow \exists\ u \in W^{2,1+\varepsilon}_{loc}$ with $\det \nabla^2 u = f$ (De-Phillipis-Figalli).
- Open problem: does there exist a $W^{1,p}$ solution of (1) for some p when f is only C^0 (and positive)?

The Prescribed Jacobian Equation: Exsitence Theory

When f is not regular enough (continuous or less):

- Non existence result (Burago-Kleiner '98, McMullen '98): for all $\varepsilon > 0$ there exists $f \in C^0(\overline{\Omega})$ with $\|f - 1\|_{L^\infty} \leq \varepsilon$ for which there exists no Lipschitz solution to (1).
- Rivière-Ye '96: $f \in C^0(\overline{\Omega})$, $f > 0$, $\Rightarrow \exists\ \phi \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$
  $f \in L^\infty(\Omega)$, $\inf f > 0$, $\Rightarrow \exists\ \phi \in \cap_{\alpha<\beta} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$ for some $\beta \leq 1$ depending on $\|f - 1\|_{L^\infty}$
- Monge-Ampère theory: $f \in C^0 \Rightarrow$, $f > 0\ \exists\ u \in \cap_{p<\infty} W^{2,p}_{loc}$ with $\det \nabla^2 u = f$ (Caffarelli)
  $f \in L^\infty$, $\inf f > 0 \Rightarrow \exists\ u \in W^{2,1+\varepsilon}_{loc}$ with $\det \nabla^2 u = f$ (De-Phillipis-Figalli).
- Open problem: does there exist a $W^{1,p}$ solution of (1) for some p when f is only C^0 (and positive)?

The Prescribed Jacobian Equation: Exsitence Theory

When f is not regular enough (continuous or less):

- Non existence result (Burago-Kleiner '98, McMullen '98): for all $\varepsilon > 0$ there exists $f \in C^0(\overline{\Omega})$ with $\|f-1\|_{L^\infty} \leq \varepsilon$ for which there exists no Lipschitz solution to (1).
- Rivière-Ye '96: $f \in C^0(\overline{\Omega})$, $f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$
  $f \in L^\infty(\Omega)$, $\inf f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<\beta} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$ for some $\beta \leq 1$ depending on $\|f-1\|_{L^\infty}$
- Monge-Ampère theory: $f \in C^0 \Rightarrow$, $f > 0$ $\exists\, u \in \cap_{p<\infty} W^{2,p}_{loc}$ with $\det \nabla^2 u = f$ (Caffarelli)
  $f \in L^\infty$, $\inf f > 0 \Rightarrow \exists\, u \in W^{2,1+\varepsilon}_{loc}$ with $\det \nabla^2 u = f$ (De-Phillipis-Figalli).
- Open problem: does there exist a $W^{1,p}$ solution of (1) for some p when f is only C^0 (and positive)?

The Prescribed Jacobian Equation: Exsitence Theory

When f is not regular enough (continuous or less):

- Non existence result (Burago-Kleiner '98, McMullen '98): for all $\varepsilon > 0$ there exists $f \in C^0(\overline{\Omega})$ with $\|f - 1\|_{L^\infty} \leq \varepsilon$ for which there exists no Lipschitz solution to (1).
- Rivière-Ye '96: $f \in C^0(\overline{\Omega})$, $f > 0$, $\Rightarrow \exists\ \phi \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega}; \overline{\Omega})$
  $f \in L^\infty(\Omega)$, $\inf f > 0$, $\Rightarrow \exists\ \phi \in \cap_{\alpha<\beta} C^{0,\alpha}(\overline{\Omega}; \overline{\Omega})$ for some $\beta \leq 1$ depending on $\|f - 1\|_{L^\infty}$
- Monge-Ampère theory: $f \in C^0 \Rightarrow$, $f > 0$ $\exists\ u \in \cap_{p<\infty} W^{2,p}_{loc}$ with $\det \nabla^2 u = f$ (Caffarelli)
  $f \in L^\infty$, $\inf f > 0 \Rightarrow \exists\ u \in W^{2,1+\varepsilon}_{loc}$ with $\det \nabla^2 u = f$ (De-Phillipis-Figalli).
- Open problem: does there exist a $W^{1,p}$ solution of (1) for some p when f is only C^0 (and positive)?

The Prescribed Jacobian Equation: Exsitence Theory

When f is not regular enough (continuous or less):

- Non existence result (Burago-Kleiner '98, McMullen '98): for all $\varepsilon > 0$ there exists $f \in C^0(\overline{\Omega})$ with $\|f - 1\|_{L^\infty} \leq \varepsilon$ for which there exists no Lipschitz solution to (1).
- Rivière-Ye '96: $f \in C^0(\overline{\Omega})$, $f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<1} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$
  $f \in L^\infty(\Omega)$, $\inf f > 0$, $\Rightarrow \exists\, \phi \in \cap_{\alpha<\beta} C^{0,\alpha}(\overline{\Omega};\overline{\Omega})$ for some $\beta \leq 1$ depending on $\|f - 1\|_{L^\infty}$
- Monge-Ampère theory: $f \in C^0 \Rightarrow$, $f > 0$ $\exists\, u \in \cap_{p<\infty} W^{2,p}_{loc}$ with $\det \nabla^2 u = f$ (Caffarelli)
  $f \in L^\infty$, $\inf f > 0 \Rightarrow \exists\, u \in W^{2,1+\varepsilon}_{loc}$ with $\det \nabla^2 u = f$ (De-Phillipis-Figalli).
- Open problem: does there exist a $W^{1,p}$ solution of (1) for some p when f is only C^0 (and positive)?

The Prescribed Jacobian Inequality

$$\begin{cases} \det \nabla\phi \geq f & \text{(a.e.) in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega. \end{cases} \tag{2}$$

- Natural necessary condition:

$$\int_\Omega f < |\Omega|.$$

- Note that if $\int_\Omega f = |\Omega|$ then (2) reduced to (1).

The Prescribed Jacobian Inequality

$$\begin{cases} \det \nabla \phi \geq f & \text{(a.e.) in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega. \end{cases} \tag{2}$$

- Natural necessary condition:

$$\int_\Omega f < |\Omega|.$$

- Note that if $\int_\Omega f = |\Omega|$ then (2) reduced to (1).

The Prescribed Jacobian Inequality

$$\begin{cases} \det \nabla \phi \geq f & \text{(a.e.) in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega. \end{cases} \tag{2}$$

- Natural necessary condition:

$$\int_\Omega f < |\Omega|.$$

- Note that if $\int_\Omega f = |\Omega|$ then (2) reduced to (1).

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^∞ case)

Assume

- $\Omega \subset \mathbb{R}^2$ *connected bounded and smooth*
- $f \in L^\infty(\Omega)$, $f \geq 0$, $\int_\Omega f < |\Omega|$.

Then there exists $\phi : \overline{\Omega} \to \overline{\Omega}$ bi-Lipschitz satisfying (2). Moreover the regularity is sharp in general.

- The case $f \in C^0$ is trivial: by convolution find $\tilde{f} \in C^\infty(\overline{\Omega})$ with $\tilde{f} \geq f$ and $\int_\Omega \tilde{f} = |\Omega|$ then apply one of the previous mentioned results.
- When $f \in L^\infty$ not longer easy: take $f = 2\chi_A$ where $A \subset \Omega$ is open and dense with $|A|$ small enough. Then there is no $\tilde{f}$ continuous with $\tilde{f} \geq f$ and $\int_\Omega \tilde{f} = |\Omega|$ (if it were the case then $\tilde{f} \geq 2$ in Ω).

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^∞ case)

Assume

- $\Omega \subset \mathbb{R}^2$ *connected bounded and smooth*
- $f \in L^\infty(\Omega)$, $f \geq 0$, $\int_\Omega f < |\Omega|$.

Then there exists $\phi : \overline{\Omega} \to \overline{\Omega}$ bi-Lipschitz satisfying (2). Moreover the regularity is sharp in general.

- The case $f \in C^0$ is trivial: by convolution find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ then apply one of the previous mentioned results.
- When $f \in L^\infty$ not longer easy: take $f = 2\chi_A$ where $A \subset \Omega$ is open and dense with $|A|$ small enough. Then there is no $\widetilde{f}$ continuous with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ (if it were the case then $\widetilde{f} \geq 2$ in Ω).

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^∞ case)

Assume

- $\Omega \subset \mathbb{R}^2$ *connected bounded and smooth*
- $f \in L^\infty(\Omega)$, $f \geq 0$, $\int_\Omega f < |\Omega|$.

Then there exists $\phi : \overline{\Omega} \to \overline{\Omega}$ bi-Lipschitz satisfying (2). Moreover the regularity is sharp in general.

- The case $f \in C^0$ is trivial: by convolution find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ then apply one of the previous mentioned results.
- When $f \in L^\infty$ not longer easy: take $f = 2\chi_A$ where $A \subset \Omega$ is open and dense with $|A|$ small enough. Then there is no $\widetilde{f}$ continuous with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ (if it were the case then $\widetilde{f} \geq 2$ in Ω).

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^∞ case)

Assume

- $\Omega \subset \mathbb{R}^2$ *connected bounded and smooth*
- $f \in L^\infty(\Omega)$, $f \geq 0$, $\int_\Omega f < |\Omega|$.

Then there exists $\phi : \overline{\Omega} \to \overline{\Omega}$ bi-Lipschitz satisfying (2). Moreover the regularity is sharp in general.

- The case $f \in C^0$ is trivial: by convolution find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ then apply one of the previous mentioned results.
- When $f \in L^\infty$ not longer easy: take $f = 2\chi_A$ where $A \subset \Omega$ is open and dense with $|A|$ small enough. Then there is no $\widetilde{f}$ continuous with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ (if it were the case then $\widetilde{f} \geq 2$ in Ω).

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^∞ case)

Assume

- $\Omega \subset \mathbb{R}^2$ *connected bounded and smooth*
- $f \in L^\infty(\Omega)$, $f \geq 0$, $\int_\Omega f < |\Omega|$.

Then there exists $\phi : \overline{\Omega} \to \overline{\Omega}$ bi-Lipschitz satisfying (2). Moreover the regularity is sharp in general.

- The case $f \in C^0$ is trivial: by convolution find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ then apply one of the previous mentioned results.
- When $f \in L^\infty$ not longer easy: take $f = 2\chi_A$ where $A \subset \Omega$ is open and dense with $|A|$ small enough. Then there is no $\widetilde{f}$ continuous with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ (if it were the case then $\widetilde{f} \geq 2$ in Ω).

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^∞ case)

Assume

- $\Omega \subset \mathbb{R}^2$ *connected bounded and smooth*
- $f \in L^\infty(\Omega)$, $f \geq 0$, $\int_\Omega f < |\Omega|$.

Then there exists $\phi : \overline{\Omega} \to \overline{\Omega}$ bi-Lipschitz satisfying (2). Moreover the regularity is sharp in general.

- The case $f \in C^0$ is trivial: by convolution find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ then apply one of the previous mentioned results.
- When $f \in L^\infty$ not longer easy: take $f = 2\chi_A$ where $A \subset \Omega$ is open and dense with $|A|$ small enough. Then there is no $\widetilde{f}$ continuous with $\widetilde{f} \geq f$ and $\int_\Omega \widetilde{f} = |\Omega|$ (if it were the case then $\widetilde{f} \geq 2$ in Ω).

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^p case)

Let $\Omega \subset \mathbb{R}^2$ connected bounded and smooth open set. Then there exists a constant $D > 2$ with the following property:

- *for every $p > 2D$*
- *for every $f \in L^p(\Omega)$ with $f \geq 0$ and $\int_\Omega f < |\Omega|$*

there exists a bi-Sobolev map ϕ with $\phi, \phi^{-1} \in W^{1,p/D}(\Omega; \Omega)$ satisfying (2).

Moreover the constant D is computable.

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^p case)

Let $\Omega \subset \mathbb{R}^2$ connected bounded and smooth open set. Then there exists a constant $D > 2$ with the following property:

- *for every $p > 2D$*
- *for every $f \in L^p(\Omega)$ with $f \geq 0$ and $\int_\Omega f < |\Omega|$*

there exists a bi-Sobolev map ϕ with $\phi, \phi^{-1} \in W^{1,p/D}(\Omega;\Omega)$ satisfying (2).

Moreover the constant D is computable.

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^p case)

Let $\Omega \subset \mathbb{R}^2$ connected bounded and smooth open set. Then there exists a constant $D > 2$ with the following property:

- *for every $p > 2D$*
- *for every $f \in L^p(\Omega)$ with $f \geq 0$ and $\int_\Omega f < |\Omega|$*

there exists a bi-Sobolev map ϕ with $\phi, \phi^{-1} \in W^{1,p/D}(\Omega;\Omega)$ satisfying (2).

Moreover the constant D is computable.

The Prescribed Jacobian Inequality: Existence of Solutions

Theorem (J. Fischer and K. '14: the L^p case)

Let $\Omega \subset \mathbb{R}^2$ connected bounded and smooth open set. Then there exists a constant $D > 2$ with the following property:

- *for every $p > 2D$*
- *for every $f \in L^p(\Omega)$ with $f \geq 0$ and $\int_\Omega f < |\Omega|$*

there exists a bi-Sobolev map ϕ with $\phi, \phi^{-1} \in W^{1,p/D}(\Omega;\Omega)$ satisfying (2).

Moreover the constant D is computable.

Functionals in Nonlinear Elasticity

Consider model functionals of the form

$$\mathscr{F}[u] := \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta} \, dx$$

where $\Omega \subset \mathbb{R}^2$ smooth and bounded, $\beta > 0$ and $\mu \geq 0$.

- Classical functionals: $\mu = 0$ (blow up when $\det \nabla u = 0$)
- However $\mu > 0$ reasonable: in practice compression beyond a certain limit (almost) impossible
- Necessary conditions for minimizers with a Dirichlet condition?

Functionals in Nonlinear Elasticity

Consider model functionals of the form

$$\mathscr{F}[u] := \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta} \, dx$$

where $\Omega \subset \mathbb{R}^2$ smooth and bounded, $\beta > 0$ and $\mu \geq 0$.

- Classical functionals: $\mu = 0$ (blow up when $\det \nabla u = 0$)
- However $\mu > 0$ reasonable: in practice compression beyond a certain limit (almost) impossible
- Necessary conditions for minimizers with a Dirichlet condition?

Functionals in Nonlinear Elasticity

Consider model functionals of the form

$$\mathscr{F}[u] := \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta} \, dx$$

where $\Omega \subset \mathbb{R}^2$ smooth and bounded, $\beta > 0$ and $\mu \geq 0$.

- Classical functionals: $\mu = 0$ (blow up when $\det \nabla u = 0$)
- However $\mu > 0$ reasonable: in practice compression beyond a certain limit (almost) impossible
- Necessary conditions for minimizers with a Dirichlet condition?

Functionals in Nonlinear Elasticity

Consider model functionals of the form

$$\mathscr{F}[u] := \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta}\, dx$$

where $\Omega \subset \mathbb{R}^2$ smooth and bounded, $\beta > 0$ and $\mu \geq 0$.

- Classical functionals: $\mu = 0$ (blow up when $\det \nabla u = 0$)
- However $\mu > 0$ reasonable: in practice compression beyond a certain limit (almost) impossible
- Necessary conditions for minimizers with a Dirichlet condition?

Necessary Conditions for Minimizers: $\mu = 0$

$$\mathscr{F}[u] = \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u)_+^\beta}\, dx$$

- Equilibrium equation

$$\int_\Omega (2\nabla \xi(u)\, \nabla u) : \nabla u - \beta \cdot \frac{1}{(\det \nabla u)_+^\beta} \operatorname{div} \xi(u)\, dx = 0$$

for all $\xi \in C^\infty_{cpt}(\Omega)$ (Ball '76/77)

- Derivation by ansatz

$$\liminf_{\varepsilon \to 0} \frac{\mathscr{F}[(\mathrm{id} + \varepsilon \xi) \circ u] - \mathscr{F}[u]}{\varepsilon} \geq 0$$

Necessary Conditions for Minimizers: $\mu = 0$

$$\mathscr{F}[u] = \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u)_+^\beta}\, dx$$

- Equilibrium equation

$$\int_\Omega (2\nabla\xi(u)\,\nabla u) : \nabla u - \beta \cdot \frac{1}{(\det \nabla u)_+^\beta} \operatorname{div} \xi(u)\, dx = 0$$

  for all $\xi \in C^\infty_{cpt}(\Omega)$ (Ball '76/77)

- Derivation by ansatz

$$\liminf_{\varepsilon \to 0} \frac{\mathscr{F}[(\mathrm{id} + \varepsilon\xi) \circ u] - \mathscr{F}[u]}{\varepsilon} \geq 0$$

Necessary Conditions for Minimizers: $\mu > 0$

$$\mathscr{F}[u] = \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta} \, dx$$

Theorem (J. Fischer and K. '14)

The Equilibrium equation holds: i.e. for all $\xi \in C^\infty_{cpt}(\Omega)$.

$$\int_\Omega (2\nabla \xi(u) \nabla u) : \nabla u - \beta \cdot \frac{\det \nabla u}{(\det \nabla u - \mu)_+^{\beta+1}} \operatorname{div} \xi(u) \, dx = 0$$

- First difficulty: need to show $\det \nabla u \cdot (\det \nabla u - \mu)_+^{-\beta-1} \in L^1(\Omega)$
- Second difficulty:

$$\mathscr{F}[(\mathrm{id} + \varepsilon \xi) \circ u] = \int_\Omega \ldots + \frac{1}{\left(\det(\mathrm{Id} + \varepsilon \nabla \xi(u)) \det \nabla u - \mu\right)_+^\beta} \, dx$$

$\Rightarrow$ regularization required on $\{\det \nabla u \gtrsim \mu\}$

Necessary Conditions for Minimizers: $\mu > 0$

$$\mathscr{F}[u] = \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta}\, dx$$

Theorem (J. Fischer and K. '14)

The Equilibrium equation holds: i.e. for all $\xi \in C^\infty_{cpt}(\Omega)$.

$$\int_\Omega (2\nabla\xi(u)\,\nabla u) : \nabla u - \beta \cdot \frac{\det \nabla u}{(\det \nabla u - \mu)_+^{\beta+1}} \operatorname{div} \xi(u)\, dx = 0$$

- First difficulty: need to show $\det \nabla u \cdot (\det \nabla u - \mu)_+^{-\beta-1} \in L^1(\Omega)$
- Second difficulty:

$$\mathscr{F}[(\mathrm{id} + \varepsilon\xi) \circ u] = \int_\Omega \ldots + \frac{1}{\left(\det(\mathrm{Id} + \varepsilon\nabla\xi(u))\det \nabla u - \mu\right)_+^\beta}\, dx$$

$\Rightarrow$ regularization required on $\{\det \nabla u \gtrsim \mu\}$

Necessary Conditions for Minimizers: $\mu > 0$

$$\mathscr{F}[u] = \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta}\, dx$$

Theorem (J. Fischer and K. '14)

The Equilibrium equation holds: i.e. for all $\xi \in C^\infty_{cpt}(\Omega)$.

$$\int_\Omega (2\nabla\xi(u)\,\nabla u) : \nabla u - \beta \cdot \frac{\det \nabla u}{(\det \nabla u - \mu)_+^{\beta+1}} \operatorname{div} \xi(u)\, dx = 0$$

- First difficulty: need to show $\det \nabla u \cdot (\det \nabla u - \mu)_+^{-\beta-1} \in L^1(\Omega)$
- Second difficulty:

$$\mathscr{F}[(\mathrm{id} + \varepsilon\xi) \circ u] = \int_\Omega \ldots + \frac{1}{(\det(\mathrm{Id} + \varepsilon \nabla \xi(u)) \det \nabla u - \mu)_+^\beta}\, dx$$

$\Rightarrow$ regularization required on $\{\det \nabla u \gtrsim \mu\}$

Necessary Conditions for Minimizers: $\mu > 0$

$$\mathscr{F}[u] = \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta}\, dx$$

Theorem (J. Fischer and K. '14)

The Equilibrium equation holds: i.e. for all $\xi \in C^\infty_{cpt}(\Omega)$.

$$\int_\Omega (2\nabla\xi(u)\,\nabla u) : \nabla u - \beta \cdot \frac{\det \nabla u}{(\det \nabla u - \mu)_+^{\beta+1}} \operatorname{div} \xi(u)\, dx = 0$$

- First difficulty: need to show $\det \nabla u \cdot (\det \nabla u - \mu)_+^{-\beta-1} \in L^1(\Omega)$
- Second difficulty:

$$\mathscr{F}[(\mathrm{id} + \varepsilon\xi) \circ u] = \int_\Omega \ldots + \frac{1}{\big(\det(\mathrm{Id} + \varepsilon \nabla \xi(u)) \det \nabla u - \mu\big)_+^\beta}\, dx$$

$\Rightarrow$ regularization required on $\{\det \nabla u \gtrsim \mu\}$

Necessary Conditions for Minimizers: $\mu > 0$

$$\mathscr{F}[u] = \int_\Omega |\nabla u|^2 + \frac{1}{(\det \nabla u - \mu)_+^\beta}\, dx$$

Theorem (J. Fischer and K. '14)

The Equilibrium equation holds: i.e. for all $\xi \in C^\infty_{cpt}(\Omega)$.

$$\int_\Omega (2\nabla\xi(u)\,\nabla u) : \nabla u - \beta \cdot \frac{\det \nabla u}{(\det \nabla u - \mu)_+^{\beta+1}} \operatorname{div} \xi(u)\, dx = 0$$

- First difficulty: need to show $\det \nabla u \cdot (\det \nabla u - \mu)_+^{-\beta-1} \in L^1(\Omega)$
- Second difficulty:

$$\mathscr{F}[(\mathrm{id} + \varepsilon\xi) \circ u] = \int_\Omega \ldots + \frac{1}{\big(\det(\mathrm{Id} + \varepsilon \nabla \xi(u)) \det \nabla u - \mu\big)_+^\beta}\, dx$$

$\Rightarrow$ regularization required on $\{\det \nabla u \gtrsim \mu\}$

Main tool: Bi-Lipschitz Maps Stretching a Measurable Planar Set

Proposition (J. Fischer and K. '14)

For every $\tau > 0$ and for every measurable set $M \subset \Omega \subset \mathbb{R}^2$ (with small enough measure with respect to τ) there exists a bi-Lipschitz map $\phi = \phi_{\tau,M} : \overline{\Omega} \to \overline{\Omega}$ preserving the boundary pointwise with

$$\begin{aligned} \det \nabla\phi &\geq 1+\tau && \text{a.e. in } M, \\ \det \nabla\phi &\geq 1 - C\sqrt{|M|}\tau && \text{a.e. in } \Omega \setminus M, \\ \|\nabla\phi - \mathrm{Id}\|_{L^p(\Omega)} &\leq C|M|^{1/(2p)}\tau && \text{for } 1 \leq p \leq \infty, \end{aligned}$$

where C is a constant depending only on Ω.

Main tool: Bi-Lipschitz Maps Stretching a Measurable Planar Set

Proposition (J. Fischer and K. '14)

For every $\tau > 0$ *and for every measurable set* $M \subset \Omega \subset \mathbb{R}^2$ *(with small enough measure with respect to* τ*) there exists a bi-Lipschitz map* $\phi = \phi_{\tau,M} : \overline{\Omega} \to \overline{\Omega}$ *preserving the boundary pointwise with*

$$\begin{aligned} \det \nabla\phi &\geq 1+\tau && \text{a.e. in } M, \\ \det \nabla\phi &\geq 1 - C\sqrt{|M|}\tau && \text{a.e. in } \Omega \setminus M, \\ \|\nabla\phi - \mathrm{Id}\|_{L^p(\Omega)} &\leq C|M|^{1/(2p)}\tau && \text{for } 1 \leq p \leq \infty, \end{aligned}$$

where C *is a constant depending only on* Ω.

Making use of those stretching maps

- For finding a bi-Lipschitz map ϕ satisfying (for $f \in L^\infty$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla \phi \geq f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega \end{cases}$$

- first find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\int_\Omega \widetilde{f} = |\Omega|$ and $|\{\widetilde{f} < f\}| << 1$
- find $\varphi \in C^\infty(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} \det \nabla \varphi = \widetilde{f} & \text{in } \Omega \\ \varphi = \text{id} & \text{on } \partial\Omega \end{cases}$$

- postcompose φ by a map streching (by a sufficiently big factor τ) the set $\varphi(\{\widetilde{f} < f\})$.

Making use of those stretching maps

- For finding a bi-Lipschitz map ϕ satisfying (for $f \in L^\infty$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla \phi \geq f & \text{in } \Omega \\ \phi = \mathrm{id} & \text{on } \partial\Omega \end{cases}$$

  - first find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\int_\Omega \widetilde{f} = |\Omega|$ and $|\{\widetilde{f} < f\}| << 1$
  - find $\varphi \in C^\infty(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} \det \nabla \varphi = \widetilde{f} & \text{in } \Omega \\ \varphi = \mathrm{id} & \text{on } \partial\Omega \end{cases}$$

  - postcompose φ by a map streching (by a sufficiently big factor τ) the set $\varphi(\{\widetilde{f} < f\})$.

Making use of those stretching maps

- For finding a bi-Lipschitz map ϕ satisfying (for $f \in L^\infty$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla\phi \geq f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega \end{cases}$$

  - first find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\int_\Omega \widetilde{f} = |\Omega|$ and $|\{\widetilde{f} < f\}| << 1$
  - find $\varphi \in C^\infty(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} \det \nabla\varphi = \widetilde{f} & \text{in } \Omega \\ \varphi = \text{id} & \text{on } \partial\Omega \end{cases}$$

  - postcompose φ by a map streching (by a sufficiently big factor τ) the set $\varphi(\{\widetilde{f} < f\})$.

Making use of those stretching maps

- For finding a bi-Lipschitz map ϕ satisfying (for $f \in L^\infty$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla \phi \geq f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega \end{cases}$$

  - first find $\widetilde{f} \in C^\infty(\overline{\Omega})$ with $\int_\Omega \widetilde{f} = |\Omega|$ and $|\{\widetilde{f} < f\}| << 1$
  - find $\varphi \in C^\infty(\overline{\Omega}; \overline{\Omega})$ satisfying

$$\begin{cases} \det \nabla \varphi = \widetilde{f} & \text{in } \Omega \\ \varphi = \text{id} & \text{on } \partial\Omega \end{cases}$$

  - postcompose φ by a map streching (by a sufficiently big factor τ) the set $\varphi(\{\widetilde{f} < f\})$.

Making use of those stretching maps

- For finding a $W^{1,p/D}$ map ϕ satisfying (for $f \in L^p$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla \phi \geq f & \text{in } \Omega \\ \phi = \mathrm{id} & \text{on } \partial\Omega: \end{cases}$$

  - basic idea: stretch the superlevel sets of f
  - more precisely: by induction construct the map ϕ_i stretching the set $\phi_{i-1} \circ \cdots \circ \phi_1(\{f \geq 2^i\})$ by a factor of 2
  - compose those maps and obtain $\phi = \lim_{i \to \infty} \phi_i \circ \cdots \circ \phi_1$.

Making use of those stretching maps

- For finding a $W^{1,p/D}$ map ϕ satisfying (for $f \in L^p$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla\phi \geq f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega: \end{cases}$$

  - basic idea: stretch the superlevel sets of f
  - more precisely: by induction construct the map ϕ_i stretching the set $\phi_{i-1} \circ \cdots \circ \phi_1(\{f \geq 2^i\})$ by a factor of 2
  - compose those maps and obtain $\phi = \lim_{i\to\infty} \phi_i \circ \cdots \circ \phi_1$.

Making use of those stretching maps

- For finding a $W^{1,p/D}$ map ϕ satisfying (for $f \in L^p$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla\phi \geq f & \text{in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega: \end{cases}$$

  - basic idea: stretch the superlevel sets of f
  - more precisely: by induction construct the map ϕ_i stretching the set $\phi_{i-1} \circ \cdots \circ \phi_1(\{f \geq 2^i\})$ by a factor of 2
  - compose those maps and obtain $\phi = \lim_{i\to\infty} \phi_i \circ \cdots \circ \phi_1$.

Making use of those stretching maps

- For finding a $W^{1,p/D}$ map ϕ satisfying (for $f \in L^p$, $f \geq 0$ and $\int_\Omega f < |\Omega|$)

$$\begin{cases} \det \nabla \phi \geq f & \text{in } \Omega \\ \phi = \mathrm{id} & \text{on } \partial\Omega: \end{cases}$$

  - basic idea: stretch the superlevel sets of f
  - more precisely: by induction construct the map ϕ_i stretching the set $\phi_{i-1} \circ \cdots \circ \phi_1(\{f \geq 2^i\})$ by a factor of 2
  - compose those maps and obtain $\phi = \lim_{i\to\infty} \phi_i \circ \cdots \circ \phi_1$.

Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0,1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0,1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0,1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0,1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0,1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0,1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0,1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0,1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0,1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0,1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0, 1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0, 1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

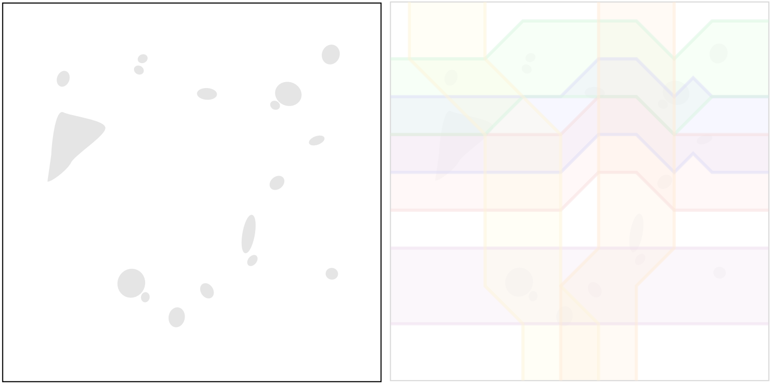
Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0,1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0,1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

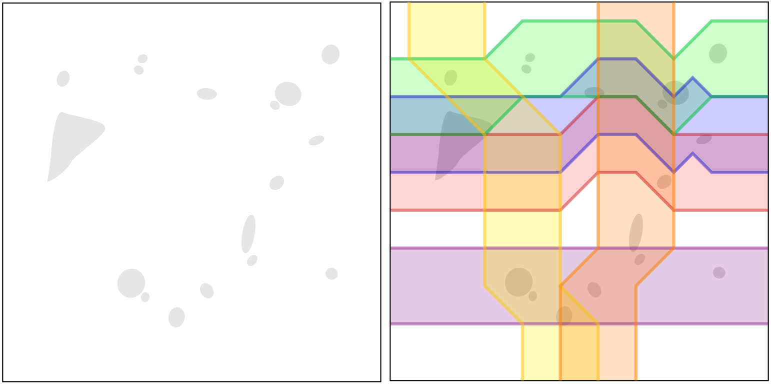
Construction of the Stretching Maps

- Simplification: with no loss of generality we can assume
  - $\Omega = (0,1)^2$
  - the set M is compact
  - ϕ presearves the boundary globally (and not pointwise)
- make use of Alberti-Csörnyei-Preiss covering: any compact set $M \subset (0,1)^2$ can be covered with by horizontal and vertical 1-Lipschitz strips with:
  - Total area of strips $\leq C\sqrt{|M|}$
  - number of intersections of horizontal (and respectively vertical) strips controlled uniformly
- Stretch the strips by a factor τ (explicit formula).

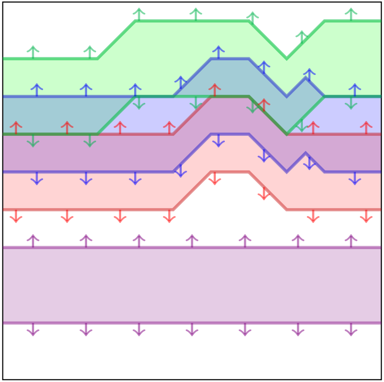
Construction of the Stretching Maps

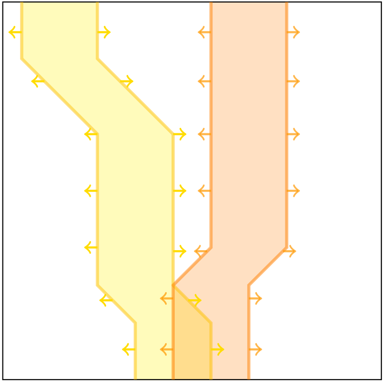


Construction of the Stretching Maps



Construction of the Stretching Maps





Thank you for your attention

Literature:

- J.F. and Olivier Kneuss, *Bi-Lipschitz Solutions to the Prescribed Jacobian Inequality in the Plane and Applications to Nonlinear Elasticity,* submitted, 2014