Variational Formula for First Passage Percolation

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Fields Institute, Toronto.
(work done in the Courant Institute, New York)

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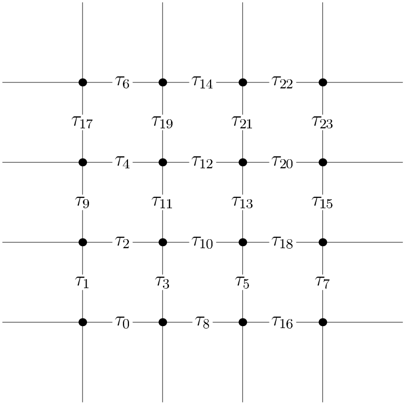
First Passage Percolation on the Lattice



- Positive random edge-weights on nearest-neighbour graph on $\mathbb{Z}^d$.

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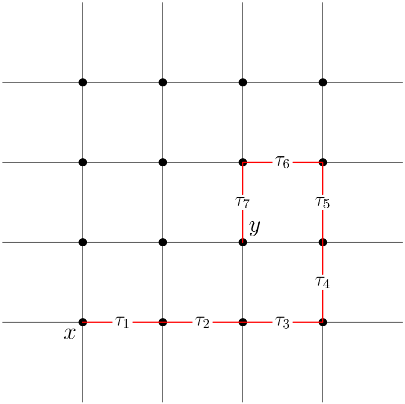

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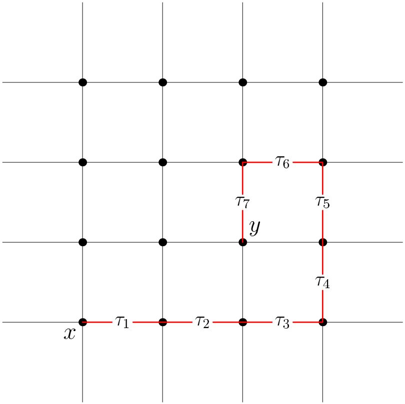

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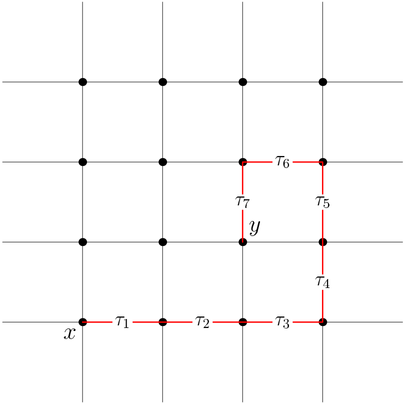

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- First-Passage Time:

$$T(x, y) = \inf_{\gamma} W(\gamma(x, y))$$

- Will write $T(x)$ for $T(x, 0)$ in general

What do we want to compute?

Time-constant $g(x)$

- Fix $x \in \mathbb{R}^d$, consider an "average" time to travel in direction x.
$$T_n(x) = \frac{T([nx])}{n}$$

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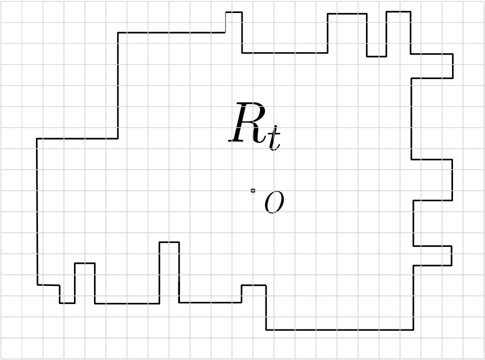
- $g(x)$ is called <u>time-constant</u>.

Motivation: the limit-shape

Consider sites occupied by time t:

$$R_t := \{x \in \mathbb{R}^d \mid T([x]) \leq t\},$$

We're interested in the limiting behavior of this set.



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Theorem [Cox and Durrett, 1981]

$$\lim_{t \to \infty} R_t / t = \{x : g(x) \leq 1\}$$

What do we prove?

Time constant solves a PDE

- Movement of light in a medium: Eikonal equation.

$$c(x)|Du(x)| = 1, \quad u(0) = 0$$

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- $g(x)$ is a norm on $\mathbb{R}^d$
- By convex duality $H(p)$ is the dual norm:

$$H(p) = \sup_{g(x)=1} x \cdot p$$

Notation for edge-weights

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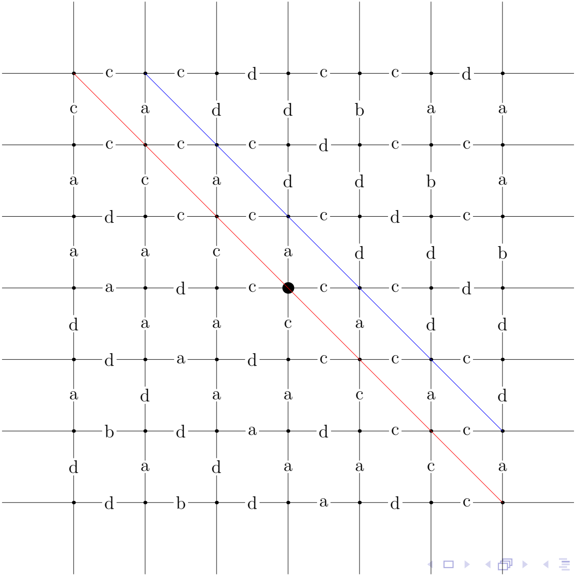
- $\tau(z, \alpha, \cdot)$ represents edge-weight at $z \in \mathbb{Z}^d$ in the $\alpha \in A$ direction
- Weights are stationary and ergodic (e.g. i.i.d.), and they're uniformly bounded (away from 0 and from above)

Assume symmetry in the medium (only for the examples)

$\tau(x, \alpha, \omega) \in \{a, b, c, d\}, \ \alpha \in \{\pm e_1, \pm e_2\}$

$\tau(\cdot, \cdot, \omega)$ is constant along $x + y = z$.



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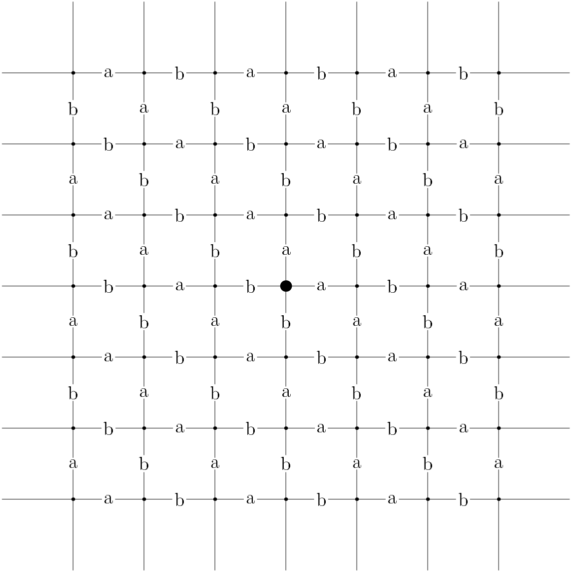
- Will play around with edge-weight marginals; all supported on $[1, 2]$. All will have $E[\tau] = 1.5$.
- Will see the level sets $\{p \in \mathbb{R}^2 : H(p) = 1\}$.
- The "bigger" the Hamiltonian level-set, the slower the percolation. It's a speed-time duality.

Example: Periodic Medium

$\tau(\cdot, \cdot, \omega) \in \{a, b\}, a < b$



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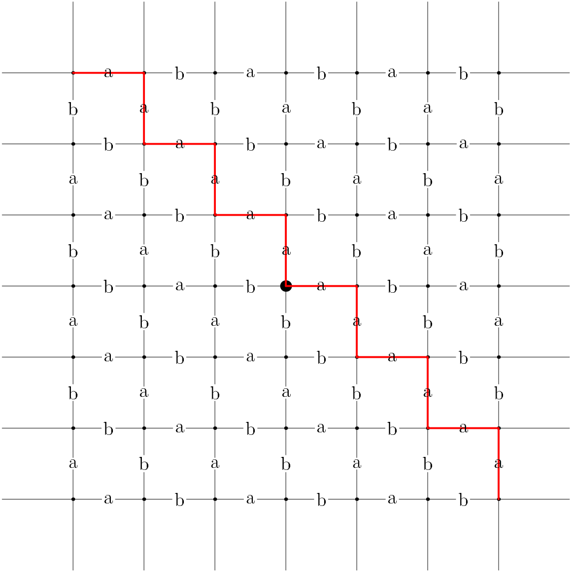
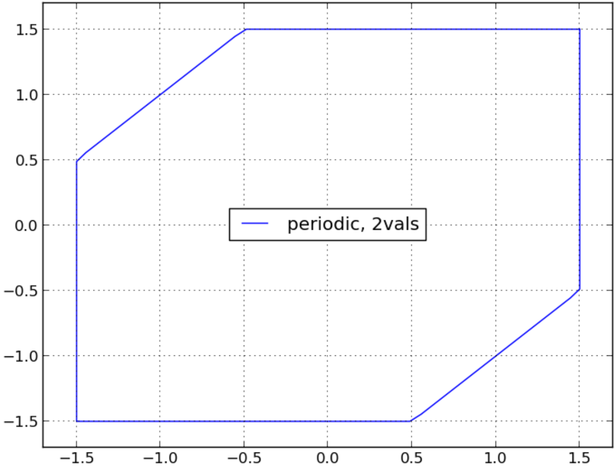


Limit Shape: Periodic Medium

$\tau \in \{1, 2\}$, Plot of $H(p) = 1$



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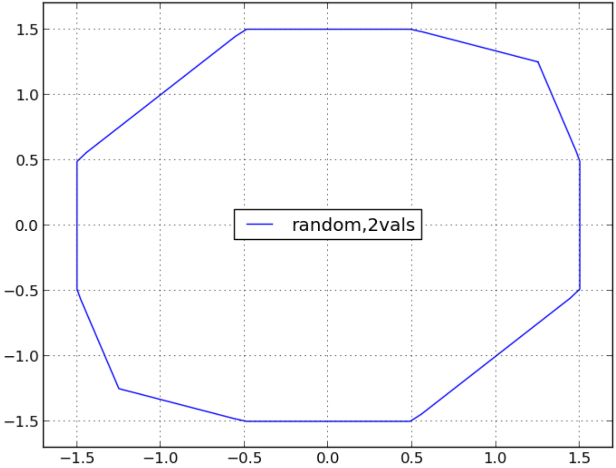
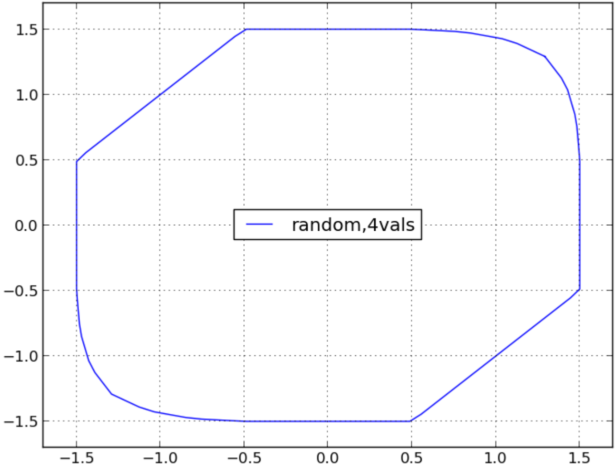


Limit Shape: Comparing different media

$\tau \in \{1, 1.33, 1.66, 2\}$, uniform measure, plot of $H(p) = 1$



Limit Shape: Comparing different media

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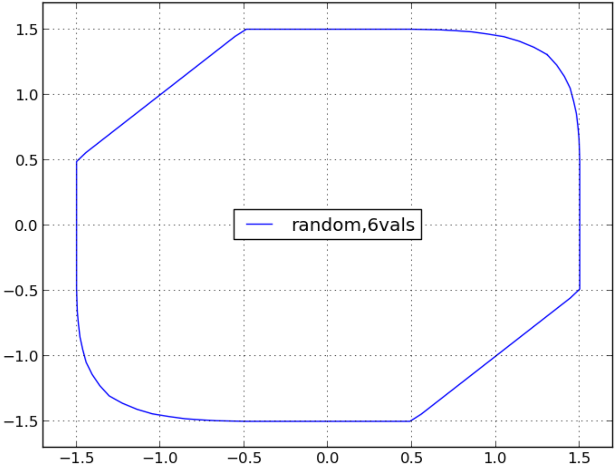
$\tau \in \{1, 1.2, 1.4, 1.6, 1.8, 2\}$, uniform measure, plot of $H(p) = 1$



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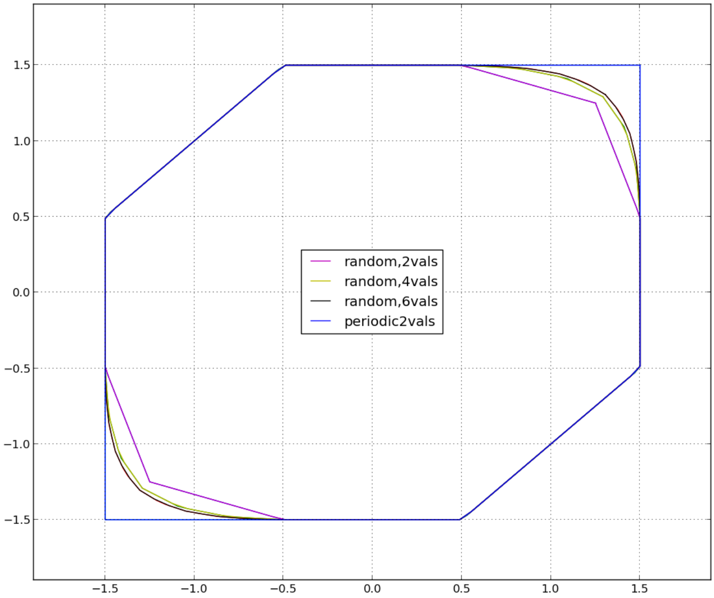

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- Future work/other applications

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- Exact limit shapes can be calculated for two special edge-weight distributions Johansson [2000], Seppäläinen [1998].
- KPZ scaling and fluctuations (in $d = 2$):

$$T([nx]) \sim g(x)n + n^{1/3}\xi$$

ξ is a random variable that's Tracy-Widom distributed (from random matrix theory) [Johansson, 2000]. Is it universal?

Notation for main theorem

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- Recall unit directions A, edge-weights $\tau(z, \alpha, \cdot)$
- For $f : \mathbb{Z}^d \to \mathbb{R}$, discrete derivative is $Df(x, \alpha) = f(x + \alpha) - f(x)$.
- Will optimize functions f, such that $E[Df] = 0$, Df stationary.

Main Theorem

Variational Formula

Theorem

For $p \in \mathbb{R}^d$, the dual norm of $g(x)$ is given by

$$H(p) = \inf_{f \in S} \operatorname*{ess\,sup}_{\omega \in \Omega} \mathcal{H}(Df + p, x, \omega),$$

where

$\mathcal{H}$ is the discrete Hamiltonian

S is a set of functions.

Link:algorithm

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where

$$\mathcal{H}(Df + p, x, \omega) = \sup_{\alpha \in A} \left\{ -\frac{Df(x, \alpha) + p \cdot \alpha}{\tau(x, \alpha, \omega)} \right\},$$

$$S = \left\{ f : \mathbb{Z}^d \to \mathbb{R} \mid E[Df] = 0, Df \text{ stationary} \right\}.$$

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- Replace this with a single variational problem for $H(p)$; minimization over functions
- Think of this is a nonlinear duality principle:

$$
\begin{aligned}
g(x) &= \lim_{n\to\infty} \frac{1}{n} \inf_{\text{paths}} (\text{“convex fn”}) \\
&= \sup_{f \in S} (\text{“Legendre transform”})
\end{aligned}
$$

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Exact limit-shape by iteration

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Symmetry Assumption

For each $z \in \mathbb{Z}$, assume

$$\tau(x, \cdot, \omega) = \tau(y, \cdot, \omega) \quad \forall\; x + y = z.$$

Algorithm to produce a minimizer

Theorem: constructing the minimizer

For any $f_0 \in S$, we give an explicit $\mathcal{I} : S \to S$ such that the sequence defined by

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converges to a minimizer.

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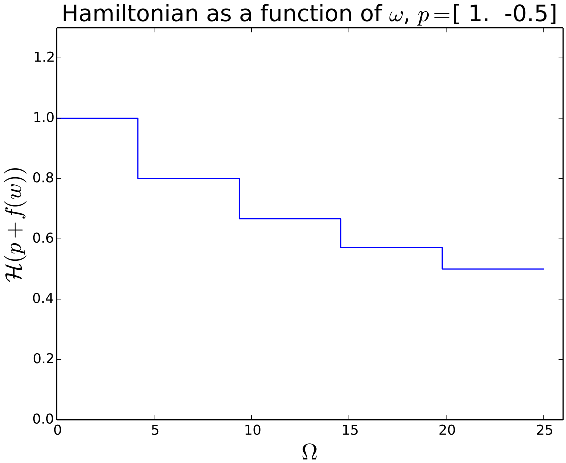
Proof implies

One of the following happens:

- Algorithm terminates in finite time at a corrector
- Algorithm terminates in finite-time at a generic minimizer
- Algorithm continues to infinity, produces corrector in limit

Algorithm in action

Show animation of algorithm in action



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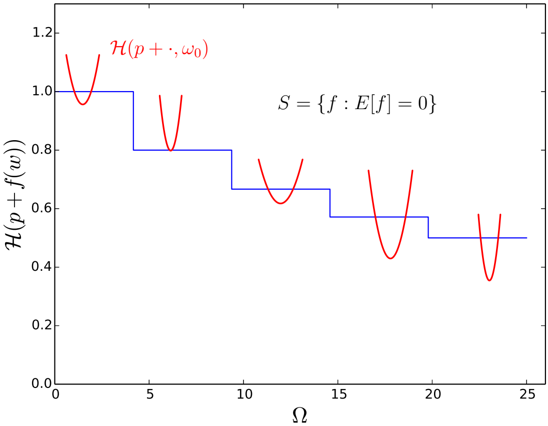

Proof sketch: where does the PDE come from?

The local characterization

- Dynamic Programming Principle:

$$T(x) = \inf_{\alpha \in A} \{ T(x+\alpha) + \tau(x,\alpha) \}.$$

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- Take a limit as $n \to \infty$, and show

$$H(Dg(x)) = 1.$$

Two different viewpoints in continuum

Viewpoint 1: Rezakhanlou and Tarver [2000], Kosygina, Rezakhanlou, and Varadhan [2006]

- Has flavor of duality principle, uses minimax theorem.
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Discrete versions

Krishnan [2013], Georgiou, Rassoul-Agha, and Seppäläinen [2013].

The cell-problem and the multiple scales ansatz

Homogenization problem

Given

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Cell problem

For fixed $p \in \mathbb{R}^d$, can you find $v(y)$ with sublinear growth such that

$$\mathcal{H}(p + Dv(y), y) = 1$$

Proof sketch: some issues

Local characterization not sufficient

Consider first-passage percolation with constant edge-weights in one dimension.

$$|T(x+1) - T(x)| = 1 \quad \forall x \in \mathbb{Z}, \quad T(0) = 0$$

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Problem

Solution is non-unique.

Proof sketch

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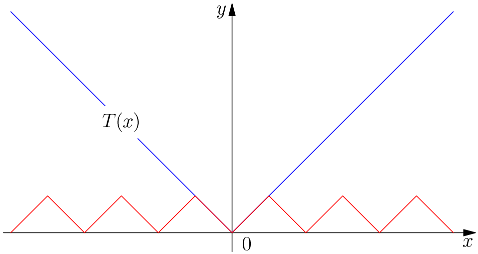

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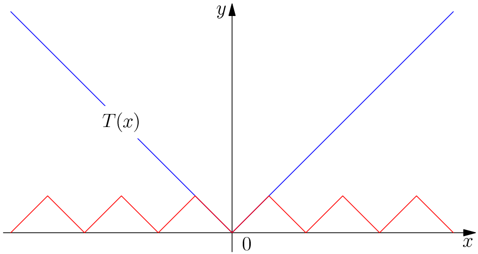
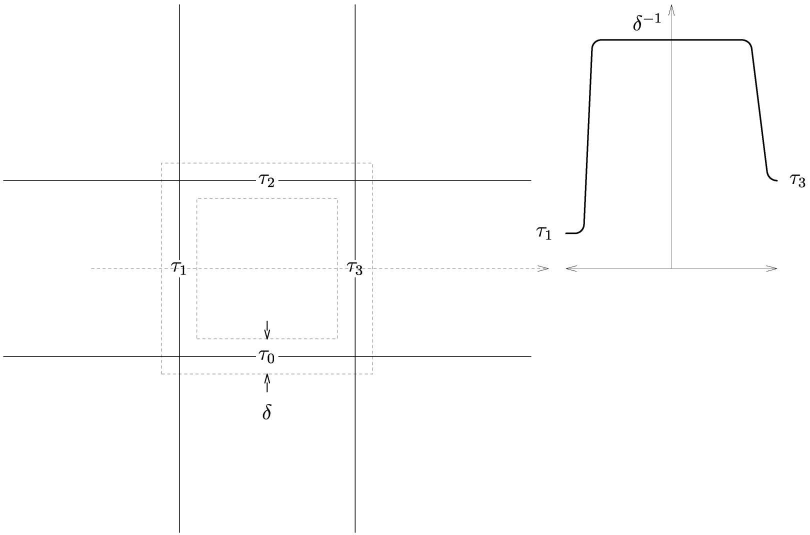


However

Solved in continuum by choosing viscosity solution.

Take problem into continuum

Make edge-weight function $\tau_\delta(x)$



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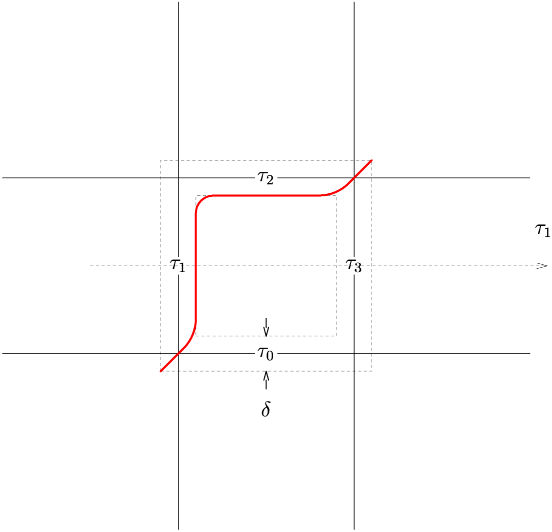

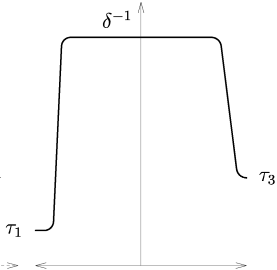

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- Strict convexity of $H(p) \Leftrightarrow$ regularity of $g(x)$.
  Use iteration to prove existence of correctors, uniqueness of minimizer and hence strict convexity of $H(p)$?
- I believe this is possible for monotone Hamiltonians (directed first-passage percolation, polymer models).

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Fluctuations

- As stated earlier, model is conjecturally in the KPZ universality class: (both scale and fluctuations)

$$T([nx]) \sim g(x)n + n^{1/3}\xi$$

ξ is Tracy-Widom distributed.

Future Work/Open Questions

Fluctuations

- As stated earlier, model is conjecturally in the KPZ universality class: (both scale and fluctuations)

$$T([nx]) \sim g(x)n + n^{1/3}\xi$$

  ξ is Tracy-Widom distributed.

- First step is to get the right scale of fluctuations (best known upper bound is $(n/\log(n))^{1/2}$ due to Benjamini et al. [2003]).

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