A quick introduction to Γ-convergence and its applications

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Outline

- Basic abstract theory
- A model case with no derivatives
- Discrete to continuum and viceversa
- Elliptic operators in divergence form
- Expansions by Γ-convergence
- Phase transitions and image segmentation
- Problems with multiple scales
- Dimension reduction
- From convergence of minimizers to evolution problems

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Introduction

The theory of Γ-convergence was invented in the '70 by E.De Giorgi.

Among the precursors of the theory, one should mention:

• the Mosco convergence (for convex functions and their duals);

• the *G*-convergence of Spagnolo for elliptic operators in divergence form;

• the epi-convergence, namely the Hausdorff convergence of the epigraphs.

But, it is only with De Giorgi and with the examples worked out by his school that the theory reached a mature stage.

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Γ-convergence is a "variational" convergence, somehow the most the natural one to pass to the limit in variational problems.

More specifically we shall deal with the Γ^- convergence, the one designed to pass to the limit in *minimum* problems.

The most general definition of Γ^- upper and lower limits, for $F : I \times X \to [-\infty, +\infty]$:

$$\begin{cases} \Gamma^{-,+} \lim F(x) := \sup_{U \ni x} \inf_{i \in I} \sup_{j \geq i} \inf_{y \in U} F(j, y), \\ \\ \Gamma^{-,-} \lim F(x) := \sup_{U \ni x} \sup_{i \in I} \inf_{j \geq i} \inf_{y \in U} F(j, y). \end{cases}$$

From now on, our index set I will be $\mathbb{N}$ and we work in a metric space (X, d), dropping the $-$ from Γ^-.

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Sequential definition of Γ-convergence

Let (X, d) be a metric space, $F_n : X \to [-\infty, +\infty]$ lower semicontinuous. As in many other cases, to define convergence we pass through the intermediate notions of upper and lower limits:

$$\Gamma - \limsup_{n\to\infty} F_n(x) := \inf\left\{\limsup_{n\to\infty} F_n(x_n) :\ x_n \to x\right\},$$

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It is obvious that $\Gamma - \liminf_n F_n \le \Gamma - \limsup_n F_n$, and it is not too difficult to check that they are both lower semicontinuous. We say that F_n Γ converge if

$$\Gamma - \limsup_{n\to\infty} F_n(x) \le \Gamma - \liminf_{n\to\infty} F_n(x) \qquad \forall x \in X$$

and we denote the common value of the upper and lower Γ limits by $\Gamma - \lim\limits_{n\to\infty} F_n$.

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How one proves Γ-convergence

As soon as we have a guess F for the Γ-limit, we have to prove that

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The first inequality means that we should be able to find $(x_n) \subset X$ convergent to x with $\limsup_n F_n(x_n) \le F(x)$. Any sequence (x_n) with this property is called *recovery* sequence.

The second inequality means that we should be able to prove, for *any* $(x_n) \subset X$ convergent to x, the lower bound for the liminf, namely $\liminf_n F_n(x_n) \ge F(x)$.

Warning!! In general pointwise convergence has nothing to do with Γ-convergence, for instance $F_n(x) = \sin(nx)$ Γ-converge to -1. In this case

$$x_n = -\frac{\pi}{2n} + \frac{2[nx/2]\pi}{n} \qquad \text{is a recovery sequence.}$$

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The two basic theorems of Γ-convergence

The first result clarifies the meaning of variational convergence: limits of (asymptotic) minimizers are minimizers and we have convergence of minimum values.

Theorem 1. *If* $\Gamma - \lim_{n\to\infty} F_n = F$ *and* $(x_n) \subset X$ *is asymptotically minimizing for* F_n*, i.e.*

$$F_n(x_n) \leq \inf_X F_n + \epsilon_n$$

with $\epsilon_n \to 0$*, then any limit point* x *of* (x_n) *minimizes* F*. In addition, under the equi-coercitivity assumption*

$$\inf_X F_n = \inf_K F_n \qquad \text{for some compact set } K \subset X \text{ independent of } n,$$

one has that F_n *attain their minimum value, and*

$$\lim_{n\to\infty} \min_X F_n = \min_X F.$$

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The two basic theorems of Γ-convergence

Proof of the first part. Let $x = \lim_{k\to\infty} x_{n(k)}$ be a limit point of (x_n). Obviously we still have $F = \Gamma - \lim_{k\to\infty} F_{n(k)}$, so that

$$\inf_X F \le F(x) \le \liminf_{k\to\infty} F_{n(k)}(x_{n(k)}) = \liminf_{k\to\infty} \inf_X F_{n(k)}.$$

On the other hand, if $(y_{n(k)})$ is a recovery sequence relative to y, then

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The two basic theorems of Γ-convergence

Proof of the first part. Let $x = \lim_{k\to\infty} x_{n(k)}$ be a limit point of (x_n). Obviously we still have $F = \Gamma - \lim_{k\to\infty} F_{n(k)}$, so that

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The two basic theorems of Γ-convergence

Theorem 2. *If (X, d) is separable, then Γ-convergence is sequentially compact.*

Proof. Let $(U_i)_{i\in\mathbb{N}}$ be a countable basis for the open sets of X, stable under finite intersections. If F_n are given, we may extract by a diagonal argument a subsequence $n(k)$ such that

$$\ell_i := \lim_{k\to\infty} \inf_{U_i} F_{n(k)} \qquad \text{exists for all } i \in \mathbb{N}.$$

Then, define

$$F(x) := \sup_{U_i \ni x} \ell_i, \qquad x \in X.$$

The Γ-liminf inequality follows by

$$\liminf_{k\to\infty} F_{n(k)}(x_k) \geq \liminf_{k\to\infty} \inf_{U_i} F_{n(k)} = \ell_i \quad \text{for all } i \text{ s.t. } x \in U_i.$$

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Other easy properties

• When the convergence is monotone, i.e. $F_n \leq F_{n+1}$, the monotone (or pointwise) limit is $F(x) = \sup_n F_n(x)$ (in this case the recovery sequence is constant). This happens, for instance for the L^p norms $\left(\int |f|^p \, d\mu\right)^{1/p}$ in a probability space, whose limit and Γ-limit as $p \uparrow \infty$ is the L^∞ norm.

• Γ-convergence is invariant under additive continuous perturbations and left compositions with non-decreasing maps:

$$F = \Gamma - \lim_{n\to\infty} F_n \qquad \Longrightarrow \qquad F + g = \Gamma - \lim_{n\to\infty} (F_n + g) \;\; \forall g \in C(X, \mathbb{R}),$$

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